Poisson paradigm

Binomial case : $np \simeq \lambda$
Here : $p_i \approx \frac{\lambda}{n}$

Situation: Consider

- n events $E_1, \ldots, E_n$ n should be large
- $p_i = \mathbf{P}(E_i)$ $\longrightarrow$ p_i should be small
- Weak dependence of the E_i: $\mathbf{P}(E_i E_j) \lesssim \frac{1}{n}$ or $P(E_i \mid E_j) \lesssim \frac{1}{n}$
- $\lim_{n\to\infty} \sum_{i=1}^{n} p_i = \lambda$

Heuristic limit: Under the conditions above we expect that

$$\#\text{ times that } E_i \text{ occurs} = X_n = \sum_{i=1}^{n} \mathbf{1}_{E_i} \to \mathcal{P}(\lambda) \tag{3}$$

$\mathbf{1}_{E_i} = \begin{cases} 1 & \text{if } E_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

Example: matching problem (1)

Situation:

- n men take off their hats
- Hats are mixed up
- Then each man selects his hat at random
- Match: if a man selects his own hat

Question: Compute

- $\mathbf{P}(E_k)$ with $E_k =$ "exactly k matches"

Random variable. Set

$X = \#$ matches

Then $\mathbb{P}(E_k) = \mathbb{P}(X = k)$

exactly k matches

Decomposition for X

$G_i \equiv$ " person i gets his own hat "

$$\Rightarrow \quad \boxed{X = \sum_{i=1}^{n} \mathbb{1}_{G_i}}$$

Summary

$X = \sum \mathbb{1}_{G_i}$, $\mathbb{P}(E_k) = \mathbb{P}(X = k)$

We wish to prove $\sum_{i=1}^{n} \frac{1}{i} \longrightarrow \infty$

(i) $p_i = \mathbb{P}(G_i) \approx \frac{1}{n}$

(ii) $\mathbb{P}(G_j \mid G_i) \approx \frac{1}{n}$ if $j \neq i$

Here

(i) $\mathbb{P}(G_i) = \mathbb{P}($person i picks his own hat among n hats$)$

$= \frac{1}{n} = p_i \Rightarrow \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{n} = 1 \equiv \lambda$

Example: matching problem (2)

Fact: Using heavy combinatorics, one can prove

$$\mathbf{P}(E_k) = \frac{1}{k!} \sum_{j=2}^{n-k} \frac{(-1)^j}{j!}$$

Thus

$$\lim_{n\to\infty} \mathbf{P}(E_k) = \frac{e^{-1}}{k!}$$

New events: We set

$$G_i = \text{"Person } i \text{ selects his own hat"}$$

Example: matching problem (3)

Probabilities for G_i: We have

$$\mathbf{P}(G_i) = \frac{1}{n}, \qquad \mathbf{P}\left(G_i \middle| G_j\right) = \frac{1}{n-1}$$

Random variable of interest:

$$X = \sum_{i=1}^{n} \mathbf{1}_{G_i} \quad \Longrightarrow \quad \mathbf{P}(E_k) = \mathbf{P}\left(X = k\right)$$

Poisson paradigm: From (3) we have $X \simeq \mathcal{P}(1)$. Therefore

$$\mathbf{P}(E_k) = \mathbf{P}\left(X = k\right) \simeq \mathbf{P}\left(\mathcal{P}(1) = k\right) = \frac{e^{-1}}{k!}$$

Outline

Geometric random variable

Notation:

$$X \sim \mathcal{G}(p), \qquad \text{for } p \in (0,1)$$

State space:

$$E = \mathbb{N} = \{1, 2, 3, \ldots\}$$

Pmf:

$$= p\, q^{k-1}$$

$$\mathbf{P}(X = k) = p\,(1-p)^{k-1}, \quad k \geq 1$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}$$

Geometric random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until first success $\Rightarrow$ $X \sim \mathcal{G}(p)$

Example: dice rolling

- Set $X = $ 1st roll for which outcome $= 6$
- We have $X \sim \mathcal{G}(1/6)$ $\Rightarrow$ $E[X] = 6$

Computing some probabilities for the example: $p(k) = (1-p)^{k-1} p$

$$\mathbf{P}(X = 5) = \left(\frac{5}{6}\right)^4 \frac{1}{6} \simeq 0.08$$

$$\mathbf{P}(X \geq 7) = \left(\frac{5}{6}\right)^6 \simeq 0.33$$

$\mathbb{E}[X]$ for $X \sim \mathcal{G}(p)$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\, p(k) = \sum_{k=1}^{\infty} k\, q^{k-1} p$$

trick

$$= \sum_{k=1}^{\infty} (k-1+1)\, q^{k-1} p \qquad = \sum_{k=1}^{\infty} p(k) = 1$$

$$= \sum_{k=1}^{\infty} (k-1)\, q^{k-1} p + \sum_{k=1}^{\infty} q^{k-1} p$$

$$= \sum_{k=1}^{\infty} (k-1)\, q^{k-2}\, q\, p + 1$$

$$= q \sum_{k=2}^{\infty} (k-1)\, q^{k-2} p + 1$$

cv: $k-1=j$

$$= q \sum_{j=1}^{\infty} j\, q^{j-1} p + 1$$

$$= q\, \mathbb{E}[X] + 1$$

Conclusion $\mathbb{E}[X]$ solves

$$\mathbb{E}[X] = q\,\mathbb{E}[X] + 1$$

$$\Rightarrow \quad \boxed{\mathbb{E}[X] = \frac{1}{1-q} = \frac{1}{p}}$$

Geometric random variable (3)

Computation of $\mathbf{E}[X]$: Set $q = 1 - p$. Then

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{i=1}^{\infty} i q^{i-1} p \\
&= \sum_{i=1}^{\infty} (i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\
&= q\, \mathbf{E}[X] + 1
\end{aligned}$$

Conclusion:

$$\mathbf{E}[X] = \frac{1}{p}$$