Poisson paradigm Binomial case: np ~ A Here : p: x =

- Situation: Consider n should be large n events E_1, \ldots, E_n or $p_i = \mathbf{P}(E_i) \longrightarrow p_i$ should be small $\mathcal{P}(E_i \mid E_j) \leq \frac{1}{n}$ Weak dependence of the E_i : $\mathbf{P}(E_i E_j) \geq \frac{1}{n}$

•
$$\lim_{n\to\infty}\sum_{i=1}^n p_i = \lambda$$

Heuristic limit: Under the conditions above we expect that

times that =
$$X_n = \sum_{i=1}^n (\mathbf{1}_{E_i}) \rightarrow \mathcal{P}(\lambda)$$
 (3)
= $\begin{cases} l & i \neq E_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

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Example: matching problem (1)

Situation:

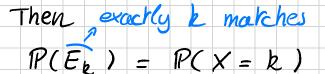
- *n* men take off their hats
- Hats are mixed up
- Then each man selects his hat at random
- Match: if a man selects his own hat

Question: Compute

•
$$\mathbf{P}(E_k)$$
 with $E_k =$ "exactly k matches"

Random variable. Set

X = # marches



Decomposition for X

Gi = " person i geb his own hat " $\Rightarrow X = \sum_{i=1}^{n} \mathbf{1}_{G_i}$

Summary X = Z 1G; P(EL)= P(X=L) we wish to prove $\sum \frac{1}{2} \rightarrow \infty$ (i) $p_i = P(G_i) \approx \frac{1}{n}$ (ii) PC GilGi) ~ th if j≠i Here (i) IP(G:) = IP(person i picks his own hat gamong n hats) $= \frac{1}{n} = \frac{p}{\epsilon} \Rightarrow \frac{2}{\epsilon_a} p = \frac{2}{\epsilon_a} = 1 = \lambda$

Example: matching problem (2)

Fact: Using heavy combinatorics, one can prove

$$\mathsf{P}(E_k) = \frac{1}{k!} \sum_{j=2}^{n-k} \frac{(-1)^j}{j!}$$

Thus

$$\lim_{n\to\infty}\mathbf{P}(E_k)=\frac{e^{-1}}{k!}$$

New events: We set

 G_i = "Person *i* selects his own hat"

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Image: A matrix

Example: matching problem (3)

Probabilities for G_i : We have

$$\mathbf{P}(G_i) = \frac{1}{n}, \qquad \mathbf{P}(G_i \mid G_j) = \frac{1}{n-1}$$

Random variable of interest:

$$X = \sum_{i=1}^{n} \mathbf{1}_{G_i} \implies \mathbf{P}(E_k) = \mathbf{P}(X = k)$$

Poisson paradigm: From (3) we have $X \simeq \mathcal{P}(1)$. Therefore

$$\mathbf{P}(E_k) = \mathbf{P}(X = k) \simeq \mathbf{P}(\mathcal{P}(1) = k) = \frac{e^{-1}}{k!}$$

Image: A matrix

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Outline

- 1 Random variables
- 2 Discrete random variables
- 3 Expected value
- Expectation of a function of a random variable
- 5 Variance
- 6 The Bernoulli and binomial random variables
- 🕖 The Poisson random variable
- Other discrete random variables
- 9 Expected value of sums of random variables
- Properties of the cumulative distribution function

Geometric random variable

Notation:

$$X \sim \mathcal{G}(p),$$
 for $p \in (0,1)$

State space:

Pmf:

$$E = \mathbb{N} = \{1, 2, 3, \ldots\}$$

$$\Rightarrow \rho q^{k-l}$$

$$P(X = k) = p (1-p)^{k-1}, \quad k \ge 1$$

Expected value and variance:

$$E[X] = \frac{1}{p},$$
 $Var(X) = \frac{1-p}{p^2}$

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Geometric random variable (2) Use:

- Independent trials, with P(success) = p
- X = # trials until first success $\Rightarrow X \sim Q(\rho)$

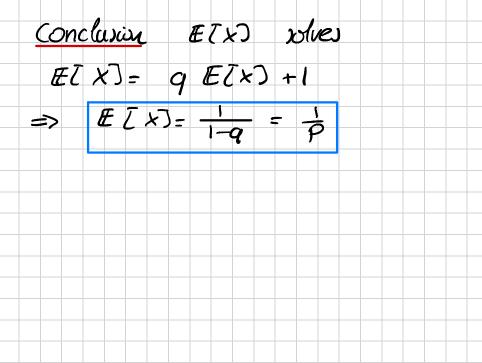
Example: dice rolling

- Set X = 1st roll for which outcome = 6
- We have $X \sim \mathcal{G}(1/6) \implies \mathbb{E}[X] = 6$

Computing some probabilities for the example: $P(k) = (1 - \rho)^{k-1} \rho$

$$P(X = 5) = \left(\frac{5}{6}\right)^4 \frac{1}{6} \simeq 0.08$$
$$P(X \ge 7) = \left(\frac{5}{6}\right)^6 \simeq 0.33$$

E(x) for X~ Q(p) $E[X] = \overset{2}{\overset{2}{\overset{}}} k \rho(k) = \overset{2}{\overset{2}{\overset{}}} k q^{k-1}$ trick (k-1+1) qk-1 $P = \frac{Z}{m}$ + Z qk-1 - 2° (k-1) 9k-1 p $= \sum_{k=1}^{\infty} (k-1) q^{k-2} q p$ + 1 q Z (k-1) qk-2 Ž j qj P + 1 E[X] + 1 q



Geometric random variable (3)

Computation of $\mathbf{E}[X]$: Set q = 1 - p. Then

$$\mathbf{E}[X] = \sum_{i=1}^{\infty} iq^{i-1}p \\ = \sum_{i=1}^{\infty} (i-1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\ = q \mathbf{E}[X] + 1$$

Conclusion:

 $\mathbf{E}[X] = \frac{1}{p}$

Samv	

Image: A matrix

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