Tail of a geometric random variable

Proposition 12.

Let

- $X \sim \mathcal{G}(p)$
- $n \geq 1$

Tail of X

Then we have

$$\mathbf{P}(X \geq n) = (1-p)^{n-1}$$

"Pf" $\mathbb{P}(X \geq n) = \sum_{k=n}^{\infty} p(k) = \sum_{k=n}^{\infty} (1-p)^{k-1} p$ ... geom. series

$= (1-p)^{n-1}$

Application Let $X \sim \mathcal{G}(\underbrace{1/6}_{p})$. Then

$$\boxed{\mathbb{P}(X \geq \underbrace{7}_{n})} = (1-p)^{n-1}$$

$$= \left(\frac{5}{6}\right)^6$$

$$\boxed{\simeq 0.33}$$

Geometric:

\# trials until 1st success

Generalization

\# trials until r-th success?

→ Negative binomial

Negative binomial random variable (1)

Notation: # success   probab. of success

$$X \sim \text{Nbin}(r, p), \text{ for } r \in \mathbb{N}^*, \; p \in (0,1)$$

$\text{Nbin}(1,p) = \mathcal{G}(p)$

State space:

$$\{r, r+1, r+2 \ldots\}$$

r-1 success during the (k-1) first trials

Pmf:

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r \, (1-p)^{k-r}, \quad k \geq r$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{r}{p}, \qquad \mathbf{Var}(X) = \frac{r(1-p)}{p^2}$$

Negative binomial random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until r successes

Justification:

$$(X = k) \\ = \\ (r-1 \text{ successes in } (k-1) \text{ 1st trials}) \cap (k\text{-th trial is a success})$$

Thus

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Moments of negative binomial random variable

Proposition 13.

Let

- $X \sim \mathrm{Nbin}(r, p)$, for $r \geq 1$, $p \in (0, 1)$
- $Y \sim \mathrm{Nbin}(r + 1, p)$
- $l \geq 1$

easiest way to compute $E[X]$, $V(X)$ ...

Then

$$\mathbf{E}\left[X^{l}\right] = \frac{r}{p}\mathbf{E}\left[(Y-1)^{l-1}\right]$$

Proof (1)

Definition of the l-th moment: We have

$$\mathbf{E}\left[X^l\right] = \sum_{k=r}^{\infty} k^l \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Relation for combination numbers:

$$k\binom{k-1}{r-1} = r\binom{k}{r}$$

Consequence:

$$\mathbf{E}\left[X^l\right] = r\sum_{k=r}^{\infty} k^{l-1} \binom{k}{r} p^r (1-p)^{k-r}$$

Proof (2)

Recall:

$$\mathbf{E}\left[X^l\right] = r \sum_{k=r}^{\infty} k^{l-1} \binom{k}{r} p^r (1-p)^{k-r}$$

From r to $r+1$:

$$\mathbf{E}\left[X^l\right] = \frac{r}{p} \sum_{k=r}^{\infty} k^{l-1} \binom{k}{(r+1)-1} p^{r+1} (1-p)^{(k+1)-(r+1)}$$

Change of variable $j = k+1$:

$$\begin{aligned}\mathbf{E}\left[X^l\right] &= \frac{r}{p} \sum_{j=r+1}^{\infty} (j-1)^{l-1} \binom{j-1}{(r+1)-1} p^{r+1} (1-p)^{j-(r+1)} \\ &= \frac{r}{p} \mathbf{E}\left[(Y-1)^{l-1}\right]\end{aligned}$$

Computation of expectation and variance

Consequence of Proposition 13:

$$\mathbf{E}[X] = \frac{r}{p}, \qquad \mathbf{Var}(X) = \frac{r(1-p)}{p^2}$$

The Banach match problem (1)

Situation:

- Pipe smoking mathematician with 2 matchboxes
- 1 box in left hand pocket, 1 box in right hand pocket
- Each time a match is needed, selected at random
- Both boxes contain initially N matches

Question:

- When one box is empty,
  what is the probability that k matches are left in the other box?

Recall: N matches in each box initially

Non symmetric event: Define

E_k = "The right box gets empty first and there are k matches in the left box"

Success: We pick the right box

Summarize: • Success = pick the right box

- Define X = # trials to reach r success

- Probab of success: p

- Initially: N matches in right box

Questions: What is # success r? → $N+1$

What is p? → $p = \frac{1}{2}$

→ We consider $X \sim NBin(N+1, \frac{1}{2})$

Summary: Consider $X \sim \mathrm{Nbin}(\underbrace{N+1}_{r}, \underbrace{\tfrac{1}{2}}_{p})$

Then

$$P(E_k) = P(X = l)$$

k matches left in the left box

$$l = N + 1 + N - k = 2N - k + 1$$

Now

$$P(X = l) = \binom{l-1}{r-1} p^r (1-p)^{l-r}$$

$$= \boxed{\binom{2N-k}{N} \underbrace{\left(\frac{1}{2}\right)^{N+1} \left(\frac{1}{2}\right)^{2N-k+1-N-1}}_{\left(\frac{1}{2}\right)^{2N-k+1}} = P(E_k)}$$

Conclusion

$\mathbb{P}($ k matches are left when 1 box is empty $)$

$= 2\ \mathbb{P}(E_k)$