Outline

cdf

Continuity of the cdf $F_X : \mathbb{R} \longrightarrow [0,1]$

Proposition 16.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$
- F the cdf of X, i.e $F(x) = \mathbf{P}(X \leq x)$

Then the function F satisfies

1. F is a nondecreasing function
2. $\lim_{b \to \infty} F(b) = 1$
3. $\lim_{b \to -\infty} F(b) = 0$ see graph later
4. F is right continuous

Proof of item 1

Inclusion property: Let $a < b$. Then

$$(X \leq a) \subset (X \leq b)$$

Consequence on probabilities:

$$\mathbf{P}(X \leq a) \leq \mathbf{P}(X \leq b)$$

Proof of item 2

Definition of an increasing sequence: Let $b_n \nearrow \infty$ and

$$E_n = (X \leq b_n)$$

Then

$$\lim_{n\to\infty} E_n = (X < \infty)$$

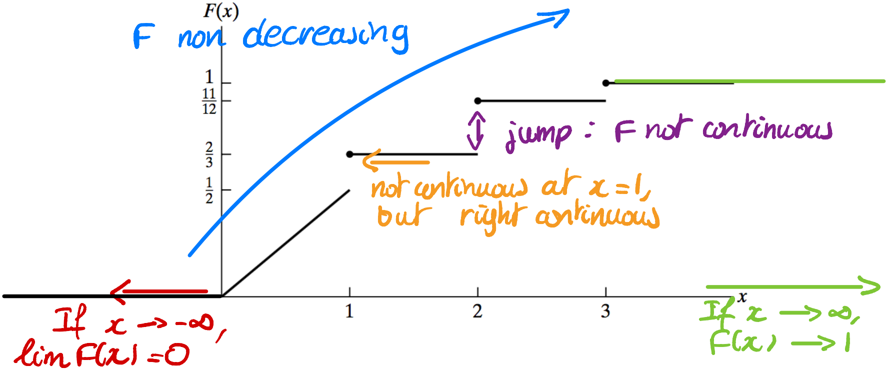
Consequence on probabilities:

$$\begin{aligned} 1 &= \mathbf{P}(X < \infty) \\ &= \mathbf{P}\left(\lim_{n\to\infty} E_n\right) \\ &= \lim_{n\to\infty} \mathbf{P}(E_n) \quad \text{(Since } n \mapsto E_n \text{ is increasing)} \\ &= \lim_{n\to\infty} F(b_n) \end{aligned}$$

Example of cdf (1)

Definition of the function: We set

$$F(x) = \frac{x}{2}\,\mathbf{1}_{[0,1)}(x) + \frac{2}{3}\,\mathbf{1}_{[1,2)}(x) + \frac{11}{12}\,\mathbf{1}_{[2,3)}(x) + \mathbf{1}_{[3,\infty)}(x)$$



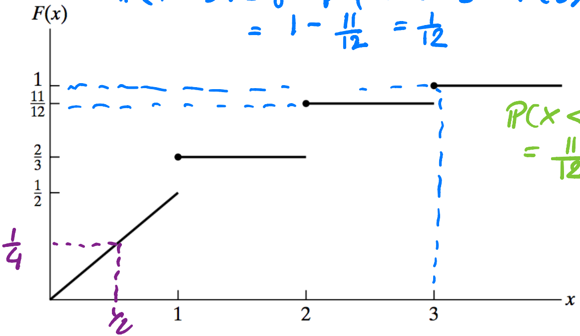
Example of cdf (2)

$F(x) = \mathbb{P}(X \le x)$

e.g. $F(3) = \mathbb{P}(X \le 3) = 1$

Some information read on the graph (see next page):

$\mathbb{P}(X=3) = \text{jump of } F \text{ at } 3 = F(3) - F(3^-)$
$= 1 - \frac{11}{12} = \frac{1}{12}$



$\mathbb{P}(X<3) = F(3^-)$
$= \frac{11}{12}$

$\mathbb{P}(X > \frac{1}{2}) = 1 - \mathbb{P}(X \le \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

$\mathbb{P}(2 < X \le 4) = \mathbb{P}(X \le 4) - \mathbb{P}(X \le 2) = F(4) - F(2)$
$= 1 - \frac{11}{12} = \frac{1}{12}$

Example of cdf (3)

Information read on the cdf: One can check that

- $\mathbf{P}(X < 3) = \frac{11}{12}$
- $\mathbf{P}(X = 1) = \frac{1}{6}$
- $\mathbf{P}(X > \frac{1}{2}) = \frac{3}{4}$
- $\mathbf{P}(2 < X \leq 4) = \frac{1}{12}$

Continuous random variables

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Mostly taken from *A first course in probability* by S. Ross

Summary of what we will see

- For discrete rv, we compute
  $\mathbb{P}(X \in \dots)$, $\mathbb{E}[X]$, ...
  with $\boxed{\text{sums}}$
- For continuous rv, we compute
  $\mathbb{P}(X \in \dots)$, $\mathbb{E}[X]$
  with $\boxed{\text{integrals}}$

Outline

General definition

Definition 1.

$\mathcal{E}$ not countable anymore

Let e.g $\mathcal{E} = [5,6]$, $\mathbb{R}$

- **P** a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$

We say that X is a continuous random variable if
$\hookrightarrow$ There exists $f \geq 0$ such that for "all" $B \subset \mathbb{R}$ we have

e.g $\mathbb{P}(10 \leq X < 15) = \int_{10}^{15} f(x)\, dx$

$$\mathbf{P}(X \in B) = \int_B f(x)\, dx$$

The function f is called

pmf in the discrete case

$\hookrightarrow$ the probability density function of the random variable X

Law of X according to f

Type of information obtained with f: We have

$$\mathbf{P}(a \le X \le b) = \int_a^b f(x)\,dx$$

$$\mathbf{P}(X = a) = 0 = \int_a^a f(x)\,dx$$

cdf F : $$F(a) = \mathbf{P}(X \le a) = \int_{-\infty}^a f(x)\,dx$$

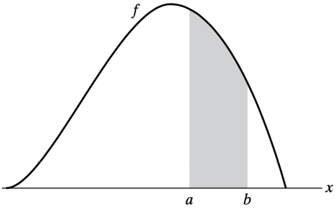
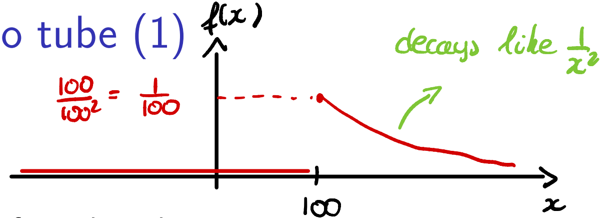


Figure: $\mathbf{P}(a \le X \le b) = \int_a^b f(x)\,dx$

Example: radio tube (1)



Situation:

- X = lifetime of a radio tube
- Density of X:

$$f(x) = \frac{100}{x^2} \mathbf{1}_{(100,\infty)}(x)$$

- We have 5 tubes in a set

Question: Probability that 2 of the 5 tubes have to be replaced within the first 150h of operation

First step: For the rv X, compute

$$\boxed{\mathbb{P}(X \leq 150)}$$

$f(x) = 0$ if $x \leq 100$

$$= \int_{-\infty}^{150} f(x)\,dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\,dx$$

$$= 100 \int_{100}^{150} \frac{1}{x^2}\,dx$$

$$= 100 \left. -\frac{1}{x} \right|_{100}^{150}$$

$$= 100 \left(\frac{1}{100} - \frac{1}{150} \right) = 1 - \frac{2}{3} \boxed{= \frac{1}{3}}$$