

Outline

- 1 Random variables
- 2 Discrete random variables
- 3 Expected value
- 4 Expectation of a function of a random variable
- 5 Variance
- 6 The Bernoulli and binomial random variables
- 7 The Poisson random variable
- 8 Other discrete random variables
- 9 Expected value of sums of random variables
- 10 Properties of the cumulative distribution function

cdf



Continuity of the cdf $F : \mathbb{R} \rightarrow [0, 1]$

Proposition 16.

Let

- \mathbf{P} a probability on a sample space S
- $X : S \rightarrow \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$
- F the cdf of X , i.e $F(x) = \mathbf{P}(X \leq x)$

Then the function F satisfies

- 1 F is a nondecreasing function
- 2 $\lim_{b \rightarrow \infty} F(b) = 1$
- 3 $\lim_{b \rightarrow -\infty} F(b) = 0$ *see graph later*
- 4 F is right continuous *↗*

Proof of item 1

Inclusion property: Let $a < b$. Then

$$(X \leq a) \subset (X \leq b)$$

Consequence on probabilities:

$$\mathbf{P}(X \leq a) \leq \mathbf{P}(X \leq b)$$

Proof of item 2

Definition of an increasing sequence: Let $b_n \nearrow \infty$ and

$$E_n = (X \leq b_n)$$

Then

$$\lim_{n \rightarrow \infty} E_n = (X < \infty)$$

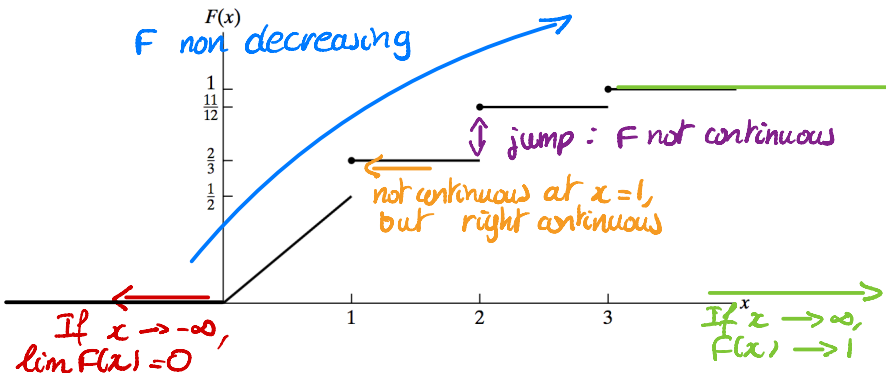
Consequence on probabilities:

$$\begin{aligned} 1 &= \mathbf{P}(X < \infty) \\ &= \mathbf{P}\left(\lim_{n \rightarrow \infty} E_n\right) \\ &= \lim_{n \rightarrow \infty} \mathbf{P}(E_n) \quad (\text{Since } n \mapsto E_n \text{ is increasing}) \\ &= \lim_{n \rightarrow \infty} F(b_n) \end{aligned}$$

Example of cdf (1)

Definition of the function: We set

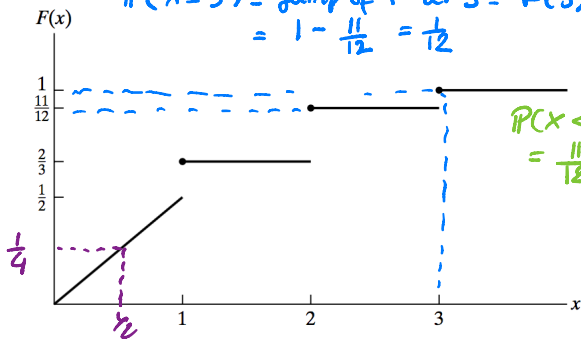
$$F(x) = \frac{x}{2} \mathbf{1}_{[0,1)}(x) + \frac{2}{3} \mathbf{1}_{[1,2)}(x) + \frac{11}{12} \mathbf{1}_{[2,3)}(x) + \mathbf{1}_{[3,\infty)}(x)$$



Example of cdf (2) $F(x) = P(X \leq x)$
 e.g. $F(3) = P(X \leq 3) = 1$

Some information read on the graph (see next page):

$$P(X=3) = \text{jump of } F \text{ at } 3 = F(3) - F(3^-) \\ = 1 - \frac{11}{12} = \frac{1}{12}$$



$$P(X > \frac{1}{2}) = 1 - P(X \leq \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(2 < X \leq 4) = P(X \leq 4) - P(X \leq 2) = F(4) - F(2) \\ = 1 - \frac{11}{12} = \frac{1}{12}$$

Example of cdf (3)

Information read on the cdf: One can check that

- $\mathbf{P}(X < 3) = \frac{11}{12}$
- $\mathbf{P}(X = 1) = \frac{1}{6}$
- $\mathbf{P}(X > \frac{1}{2}) = \frac{3}{4}$
- $\mathbf{P}(2 < X \leq 4) = \frac{1}{12}$

Continuous random variables

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Probability - MA 416

Mostly taken from *A first course in probability*
by S. Ross

Summary of what we will see

- For discrete rv, we compute $P(X \in \dots)$, $E[X]$, ...
with sums
- For continuous rv, we compute $P(X \in \dots)$, $E[X]$
with integrals

Outline

- 1 Introduction
- 2 Expectation and variance of continuous random variables
- 3 The uniform random variable
- 4 Normal random variables
- 5 Exponential random variables
- 6 Other continuous distributions
- 7 The distribution of a function of a random variable

General definition

Definition 1.

Let

e.g. $\mathcal{E} = [5, 6], \mathbb{R}$

- \mathbf{P} a probability on a sample space S
- $X : S \rightarrow \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$

\mathcal{E} not countable anymore

We say that X is a **continuous random variable** if

\Leftrightarrow There exists $f \geq 0$ such that for "all" $B \subset \mathbb{R}$ we have

e.g. $\mathbb{P}(10 \leq X < 15) = \int_{10}^{15} f(x) dx$

$$\mathbf{P}(X \in B) = \int_B f(x) dx$$

The function f is called

\nearrow pmf in the discrete case

\Leftrightarrow the probability density function of the random variable X

Law of X according to f

Type of information obtained with f : We have

$$\mathbf{P}(a \leq X \leq b) = \int_a^b f(x) dx$$

$$\mathbf{P}(X = a) = 0 = \int_a^a f(x) dx$$

cdf F : $F(a) = \mathbf{P}(X \leq a) = \int_{-\infty}^a f(x) dx$

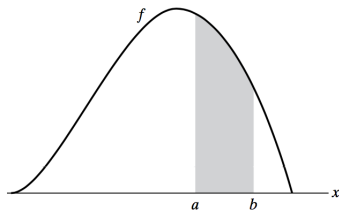
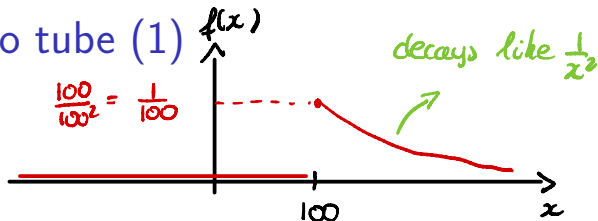


Figure: $\mathbf{P}(a \leq X \leq b) = \int_a^b f(x) dx$

Example: radio tube (1)



Situation:

- X = lifetime of a radio tube
- Density of X :

$$f(x) = \frac{100}{x^2} \mathbf{1}_{(100, \infty)}(x)$$

- We have 5 tubes in a set

Question: Probability that 2 of the 5 tubes have to be replaced within the first 150h of operation

First step: For the rv X , compute

$$P(X \leq 150) = 0 \text{ if } x \leq 100$$

$$= \int_{-\infty}^{150} f(x) dx$$

$$= \int_{100}^{150} \frac{100}{x^2} dx$$

$$= 100 \int_{100}^{150} \frac{1}{x^2} dx$$

$$= 100 \left. -\frac{1}{x} \right|_{100}^{150}$$

$$= 100 \left(\frac{1}{100} - \frac{1}{150} \right) = 1 - \frac{2}{3} = \frac{1}{3}$$