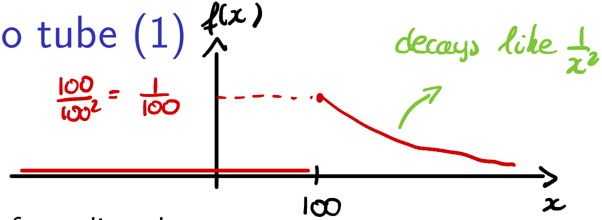
Example: radio tube (1)



Situation:

- X = lifetime of a radio tube
- Density of X:

$$f(x) = \frac{100}{x^2} \mathbf{1}_{(100,\infty)}(x)$$

- We have 5 tubes in a set

Question: Probability that 2 of the 5 tubes have to be replaced within the first 150h of operation

First step: For the rv X, compute

$$\mathbb{P}(X \leq 150) = \mathbb{P}(X \in (-\infty, 150])$$

$f(x) = 0$ if $x \leq 100$

$$= \int_{-\infty}^{150} f(x)\, dx$$

$$= \int_{100}^{150} \frac{100}{x^2}\, dx$$

$$= 100 \int_{100}^{150} \frac{1}{x^2}\, dx$$

$$= 100 \left. -\frac{1}{x} \right|_{100}^{150}$$

$$= 100 \left(\frac{1}{100} - \frac{1}{150} \right) = 1 - \frac{2}{3} = \frac{1}{3}$$

Step 2 : Take the 5 components into account

Define, for $i = 1, \dots, 5$,

$$Y_i = \begin{cases} 1 & \text{if } i\text{-th component fails in the 1st 150 h} \\ 0 & \text{otherwise} \end{cases}$$

Then Y_i is discrete. In fact

$$\boxed{Y_i \sim B\left(p = \frac{1}{3}\right)}$$

Y_i's are $\perp\!\!\!\perp$

RV of interest

Z = # components failing in the 1st 150 h

Then $Z = \sum_{i=1}^{5} Y_i \sim \text{Bin}(n=5, p=\frac{1}{3})$

We wish to compute

$$\boxed{\mathbb{P}(Z=2)} = \binom{5}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3$$

$$\boxed{\simeq 33\%}$$

Example: radio tube (2)

Family of events: We define

- X_i = lifetime of tube i
- E_i = "tube i has to be replaced within the first 150h of operation

Probability of E_i:

$$
\begin{aligned}
\mathbf{P}(E_i) &= \mathbf{P}(X_i \leq 150) \\
&= \int_{-\infty}^{150} f(x)\, dx \\
&= 100 \int_{100}^{150} \frac{dx}{x^2}
\end{aligned}
$$

Thus

$$\mathbf{P}(E_i) = \frac{1}{3}$$

Example: radio tube (3)

Model for the set of tubes: Define

$$Z_i = \mathbf{1}_{E_i}, \qquad Z = \sum_{i=1}^{5} Z_i$$

Then

$$Z \sim \text{Bin}\left(5, \frac{1}{3}\right)$$

and we look for

$$\mathbf{P}(Z = 2)$$

Conclusion:

$$\mathbf{P}(Z = 2) = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 \simeq 33\%$$

Outline

Discrete case

$$\mathbb{E}[X] = \sum_i x_i \, p(x_i)$$

Discrete → continuous

pmf p → density f

$\sum$ → $\int$

Thus

$$\boxed{\mathbb{E}[X] = \int x \, f(x) \, dx}$$

General definition

Definition 2.

Let

- **P** a probability on a sample space S
- $X : S \to \mathbb{R}$ a continuous random variable
- $f =$ density of X

Then the expected value of X is defined by

$$\mathbf{E}[X] = \int_{\mathbb{R}} x\, f(x)\, dx$$

Heuristics for the definition

Recall the discrete case:

$$\mathbf{E}[X] = \sum_{i \geq 1} x_i \, \mathbf{P}(X = x_i)$$

Continuous case analog: We have

$$f(x)\, dx \simeq \mathbf{P}(x \leq X \leq x + dx)$$

Thus

$$\begin{aligned} \mathbf{E}[X] &\simeq \sum x_i \, \mathbf{P}(x_i \leq X \leq x_i + dx) \\ &\simeq \int_{\mathbb{R}} x \, f(x) \, dx \end{aligned}$$

Simple example (1)

Density of X: Consider X with density

$$f(x) = 2x\, \mathbf{1}_{[0,1]}(x)$$

Example

$$f(x) = 2x \, \mathbb{1}_{[0,1]}(x)$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx$$

$$= \int_0^1 x \times 2x \, dx$$

$$= 2 \int_0^1 x^2 \, dx$$

$$= 2 \left. \frac{x^3}{3} \right|_0^1$$

$$= \frac{2}{3}$$

Security check If f is a density, then

(i) $f(x) \geq 0$ for all $x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (discrete: $\sum p(x_i) = 1$)

Here, with $f(x) = 2x\,\mathbb{1}_{[0,1]}(x)$,

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx} = \int_0^1 f(x)\,dx$$

$$= \int_0^1 (2x)\,dx = x^2\Big|_0^1 \boxed{= 1}$$

ok!

Simple example (2)

Recall: We consider X with density

$$f(x) = 2x\, \mathbf{1}_{[0,1]}(x)$$

Expected value:

$$\begin{aligned} \mathbf{E}[X] &= \int_{\mathbb{R}} x\, f(x)\, dx \\ &= \int_0^1 2x^2\, dx \\ &= \frac{2}{3} \end{aligned}$$

Expression for $\mathbf{E}[X]$ when $X \geq 0$

Proposition 3.

Let

- X continuous random variable
- f density of X

Hypothesis:

$f(x)=0$ if $x<0$

$$X \geq 0$$

Then

tail of X

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > y)\, dy \tag{1}$$

Proof

Expression for the rhs:

$$\int_0^\infty \mathbf{P}(X > y)\, dy = \int_0^\infty \left(\int_y^\infty f(x)\, dx \right) dy$$

Apply Fubini: Invert the order of integration

$$\begin{aligned} \int_0^\infty \mathbf{P}(X > y)\, dy &= \int_0^\infty \left(\int_0^x dy \right) f(x)\, dx \\ &= \int_0^\infty x\, f(x)\, dx \\ &= \mathbf{E}[X] \end{aligned}$$

Definition of $\mathbf{E}[g(X)]$

discrete : $E[g(X)] = \sum_i g(x_i)\, p(x_i)$

Proposition 4.

Let

- X continuous random variable
- f density of X
- g real valued function

Then

$$\mathbf{E}\left[g(X)\right] = \int_{\mathbb{R}} g(x)\, f(x)\, dx \tag{2}$$

Simple example – Ctd (1)

Density of X: Consider X with density

$$f(x) = 2x\, \mathbf{1}_{[0,1]}(x)$$

Question: Compute

$$\mathbf{E}[X^3]$$

Example - ctd

$f(x) = 2x \, \mathbb{1}_{[0,1]}(x)$

Then

$$\mathbb{E}[X^3] = \int_{-\infty}^{\infty} x^3 f(x) \, dx$$

$$= \int_0^1 x^3 (2x) \, dx$$

$$= 2 \int_0^1 x^4 \, dx$$

$$= 2 \left. \frac{x^5}{5} \right|_0^1$$

$$= \frac{2}{5}$$

Simple example – Ctd (2)

Recall: We consider X with density

$$f(x) = 2x\,\mathbf{1}_{[0,1]}(x)$$

Expected value for $g(x) = x^3$:

$$\begin{aligned} \mathbf{E}[X^3] &= \int_{\mathbb{R}} x^3 f(x)\, dx \\ &= \int_0^1 2x^4\, dx \\ &= \frac{2}{5} \end{aligned}$$

Proof of Proposition 4

Hypothesis:

We assume $X \geq 0$ and $g(X) \geq 0$ for the proof

Expression with (1):

$$\begin{aligned}
\mathbf{E}[g(X)] &= \int_0^\infty \mathbf{P}(g(X) > y)\, dy \\
&= \int_0^\infty \left(\int_{\{x;\, g(x)>y\}} f(x)\, dx \right) dy
\end{aligned}$$

Apply Fubini: Invert the order of integration

$$\begin{aligned}
\mathbf{E}[g(X)] &= \int_0^\infty \left(\int_0^{g(x)} dy \right) f(x)\, dx \\
&= \int_0^\infty g(x)\, f(x)\, dx
\end{aligned}$$

Expectation and linear transformations

Proposition 5.

Let

- X continuous random variable
- f density of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{E}[aX + b] = a\,\mathbf{E}[X] + b$$

Proof

Application of relation (2):

$$\begin{aligned} \mathbf{E}[aX+b] &= \int_{\mathbb{R}} (a\,x+b)\; f(x)\,dx \\ &= a\int_{\mathbb{R}} x\,f(x)\,dx + b\int_{\mathbb{R}} f(x)\,dx \\ &= a\,\mathbf{E}[X]+b \end{aligned}$$

Definition of variance

Definition 6.

Let

- X continuous random variable
- f density of X
- $\mu = \mathbf{E}[X]$

Then we define $\mathbf{Var}(X)$ by

$$\mathbf{Var}(X) = \mathbf{E}\left[(X-\mu)^2\right]$$

Alternative expression for the variance

Proposition 7.

Let

- X continuous random variable
- f density of X
- $\mu = \mathbf{E}[X]$

Then $\mathbf{Var}(X)$ can be written as

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - \mu^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Example - ctd $f(x) = 2x \, \mathbb{1}_{[0,1]}(x)$

Then

$$\mathbb{E}[X^2] = \int_0^1 x^2 (2x) \, dx$$

$$= 2 \int_0^1 x^3 \, dx = 2 \times \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$\boxed{Var(X)} = \underbrace{\mathbb{E}[X^2]}_{1/2} - (\underbrace{\mathbb{E}[X]}_{2/3})^2$$

$$= \frac{1}{2} - \left(\frac{2}{3}\right)^2 \boxed{= \frac{1}{18}}$$

Security check: verify that $Var(X) > 0$

Simple example – Ctd

Density of X: Consider X with density

$$f(x) = 2x\,\mathbf{1}_{[0,1]}(x)$$

Expected value for $g(x) = x^2$:

$$\mathbf{E}[X^2] = \int_{\mathbb{R}} x^2\, f(x)\, dx = \int_0^1 2x^3\, dx = \frac{1}{2}$$

Variance of X:

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Variance and linear transformations

Proposition 8.

Let

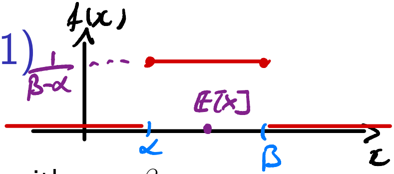
- X continuous random variable
- f density of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{Var}(aX + b) = a^2 \, \mathbf{Var}(X)$$

Outline

Uniform random variable (1)



Notation:

$$X \sim \mathcal{U}([\alpha, \beta]), \text{ with } \alpha < \beta$$

State space:

$$X \in [\alpha, \beta]$$

Density:

f is constant (uniform) on [α, β]

$$f(x) = \frac{1}{\beta - \alpha} \mathbf{1}_{[\alpha,\beta]}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{\alpha + \beta}{2}, \qquad \mathbf{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$