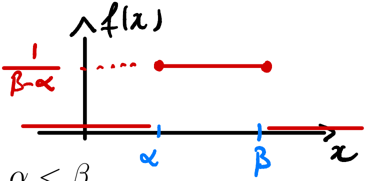
Outline

Uniform random variable (1)



Notation:

$$X \sim \mathcal{U}([\alpha, \beta]), \text{ with } \alpha < \beta$$

State space:

$$X \in [\alpha, \beta]$$

uniform
constant density
on $[\alpha, \beta]$

Density:

$$f(x) = \frac{1}{\beta - \alpha} \mathbf{1}_{[\alpha,\beta]}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{\alpha + \beta}{2}, \qquad \mathbf{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

$\mathbb{E}[X]$ for $X \sim U([\alpha, \beta])$

$$\boxed{\mathbb{E}[X]} = \int_{\mathbb{R}} x \, f(x) \, dx$$

$= \begin{cases} 1 & \text{if } x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$

$$= \int_{\mathbb{R}} x \, \frac{1}{(\beta - \alpha)} \, \overbrace{\mathbb{1}_{[\alpha,\beta]}(x)} \, dx$$

$$= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x \, dx$$

$$= \frac{1}{\beta - \alpha} \left. \frac{x^2}{2} \right|_{\alpha}^{\beta} = \frac{1}{\beta - \alpha} \, \frac{\beta^2 - \alpha^2}{2}$$

$$= \frac{1}{2} \, \frac{\cancel{(\beta - \alpha)}(\beta + \alpha)}{\cancel{\beta - \alpha}} \qquad \boxed{= \frac{\alpha + \beta}{2}}$$

Rmk about σ_X If $X \sim \mathcal{U}([\alpha, \beta])$, then

$$\sigma_X = \frac{1}{\sqrt{12}} (\beta - \alpha)$$

$\Rightarrow$ $\boxed{\text{Fluctuation of } X \text{ are proportional to } \beta - \alpha = \text{size of the interval}}$

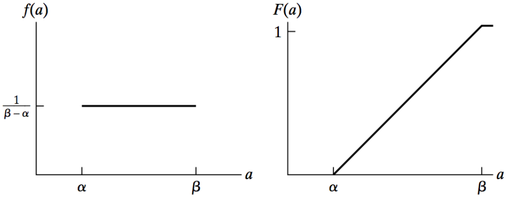
Uniform random variable (2)

Use:

- $\mathcal{U}([0,1])$ only r.v directly accessible on a computer
  $\hookrightarrow$ `rand` function → then derive all other r.v's with $\mathcal{U}([0,1])$

Example of computation: if $X \sim \mathcal{U}([8,10])$, then

$$\mathbf{P}(7.5 < X < 9.5) = \frac{1}{2}\int_{8}^{9.5} dx = \frac{9.5-8}{2} = \frac{3}{4}$$



Example: If $X \sim U(\underbrace{[8,10]}_{\text{state space for } X \text{ is } [8,10]})$, then

$$\boxed{\mathbb{P}(7.5 < X < 9.5)}$$

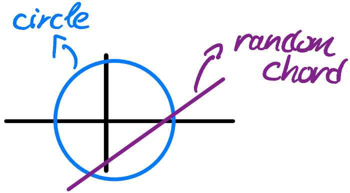
$$= \underbrace{\mathbb{P}(7.5 < X < 8)}_{=0} + \mathbb{P}(8 \leq X < 9.5)$$

$$= \int_8^{9.5} f(x)\,dx = \int_8^{9.5} \frac{1}{10-8}\,dx$$

$$= \frac{1}{2}\int_8^{9.5} dx$$

$$= \frac{1}{2}(9.5-8) \boxed{= \frac{3}{4}}$$

Bertrand's paradox (1)



Experiment:

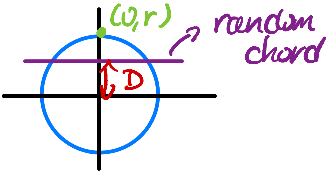
- Draw a random chord of a circle with center O and radius r

Question: Compute

$$p = \text{probability that the chord is larger than the side of the inscribed equilateral triangle}$$

Pb: unspecified model

Bertrand's paradox (2)



Model 1:

- Chord determined by its distance D to the center
- $D \sim \mathcal{U}([0,r])$

Then chord is longer than inscribed equilateral triangle iff $D \leq$ distance for equilateral triangle $= \frac{r}{2}$

Computation of p under Model 1:

$$p = \underbrace{\mathbf{P}\left(D \leq \frac{r}{2}\right)}_{} = \frac{1}{2}$$

$$= \int_0^{r/2} \frac{1}{r} \mathbb{1}_{[0,r]}(x)\, dx$$

$$= \frac{1}{r} \int_0^{r/2} dx = \frac{r}{2} \times \frac{1}{r} = \frac{1}{2}$$

Bertrand's paradox (3)

Model 2:

- Chord parametrized by θ
- $\theta =$ angle between chord and tangent
- $\theta \sim \mathcal{U}([0, 90])$

for equilateral triangle
$\theta = 60$

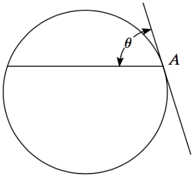
If chord larger than
equilat. triangle, then
$\theta \geq 60$

Computation of p under Model 2:

According to tangent-chord theorem

$\mathcal{U}([0, 90])$ computation

$$p = \mathbf{P}(60 < \theta < 90) = \frac{90 - 60}{90} = \frac{1}{3} \neq \frac{1}{2} \; !$$



Outline

Normal random variable (1)

Notation:

$$\mathcal{N}(\mu, \sigma^2), \text{ with } \mu \in \mathbb{R} \text{ and } \sigma^2 > 0$$

State space:

$$x \in \mathbb{R}$$

Density:

$$f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Expected value and variance:

$$\mathbf{E}[X] = \mu, \qquad \mathbf{Var}(X) = \sigma^2$$

Normal random variable (2)

Use:

Quantities which depend on a large number of small random parameters

stems from Central Limit Theorem

Numerous examples in:

- Biology
- Physics and industry
- Economics

Normal random variable (3)

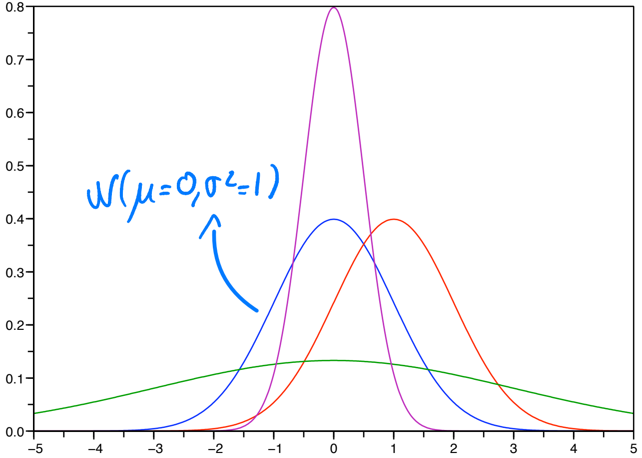


Figure: densities for $\mathcal{N}(0,1)$, $\mathcal{N}(1,1)$, $\mathcal{N}(0,9)$, $\mathcal{N}(0,1/4)$.