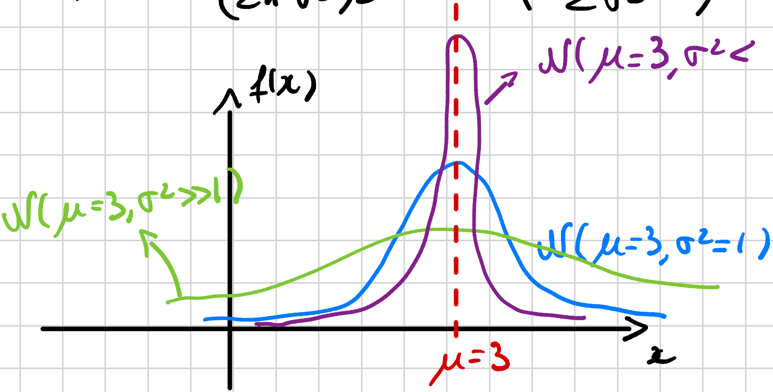
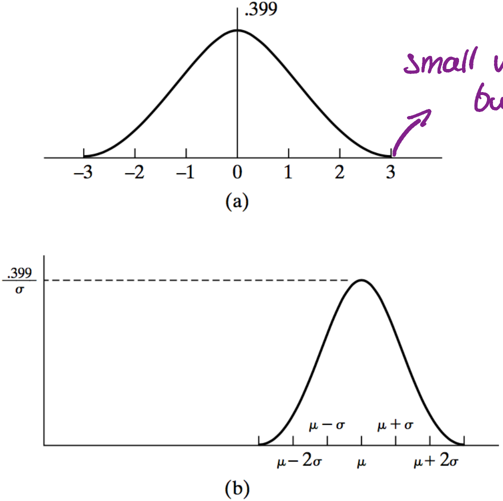
Gaussian / normal distribution

$$f(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Normal random variable (4)



small values, but not 0

Figure: Densities for (a) $\mathcal{N}(0,1)$ (b) $\mathcal{N}(\mu,\sigma^2)$

Cdf for a normal r.v

If $X \sim \mathcal{N}(0,1)$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Function Φ: For $X \sim \mathcal{N}(0,1)$ and $x \geq 0$, set

$$\Phi(x) = \underbrace{\mathbf{P}(X \leq x)}_{\text{cdf of } X} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, dy$$

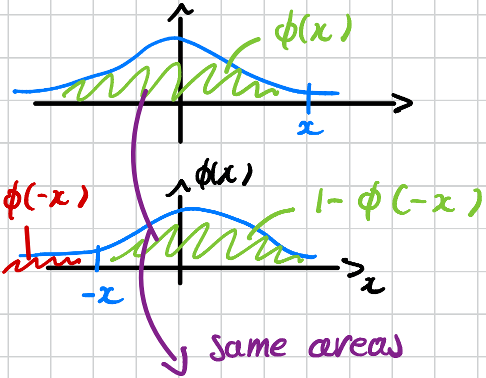
Problem with Φ:

- No algebraic expression
- Numerical approximation needed $\to$ In any decent software
- Use of tables

Property of Φ: For $x \geq 0$,

$$\Phi(-x) = 1 - \Phi(x)$$

Justification of $\phi(-x) = 1 - \phi(x)$



Thus $\phi(x) = 1 - \phi(-x)$

$$\Rightarrow \boxed{\phi(-x) = 1 - \phi(x)}$$

Table for $\Phi(x)$ for $0 \le x \le 2.49$

$\phi(1.24) = .8925$

X	.00	.01	.02	.03	$\boxed{.04}$	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
$\boxed{1.2}$	.8849	.8869	.8888	.8907	$\boxed{.8925}$	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936

Simple normal computation (1)

Definition of a random variable: We let

$$X \sim \mathcal{N}\left(\mu = 3,\, \sigma^2 = 9\right)$$

Questions: Compute

1. $\mathbf{P}(2 < X < 5)$
2. $\mathbf{P}(X > 0)$
3. $\mathbf{P}(|X - 3| > 6)$

$\mathbb{P}(2 < x < 5)$ for $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 9)$

From Prop 9:

$$Z = \frac{X-3}{\sqrt{9}} = \frac{X-3}{3} \sim \mathcal{N}(0,1)$$

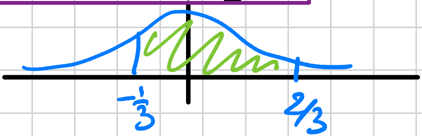
Thus

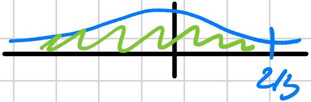
$$\mathbb{P}(2 < X < 5) = \mathbb{P}\left(\frac{2-3}{3} < \overset{= Z}{\frac{X-3}{3}} < \frac{5-3}{3}\right)$$

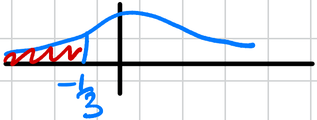
$$= \mathbb{P}\left(-\frac{1}{3} < Z < \frac{2}{3}\right)$$

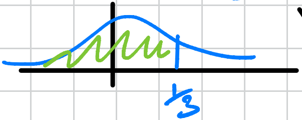
Computation of $N(0,1)$-probability

$$\boxed{\mathbb{P}\left(-\tfrac{1}{3} \le Z \le \tfrac{2}{3}\right)}$$

$=$ 

$=$ 

$-$ 

$= \Phi\left(\tfrac{2}{3}\right) - \Big(1 -$  $\Big)$

$$= \Phi\left(\tfrac{2}{3}\right) + \Phi\left(\tfrac{1}{3}\right) - 1 = .7454 + .6293 - 1$$

$$\boxed{= .3779}$$

Computation of

$$\boxed{\mathbb{P}(X > 0)} = \mathbb{P}\left(\frac{X-3}{3} > \frac{0-3}{3}\right)$$

$$= \mathbb{P}(Z > -1)$$

$$\phi(-x) = 1 - \phi(x)$$

$$= 1 - \mathbb{P}(Z \leq -1)$$

$$= 1 - (1 - \phi(1))$$

$$= \phi(1)$$

$$\boxed{= .8413}$$

Simple normal computation (2)

Change of variable: We define $Z \sim \mathcal{N}(0,1)$ by

$$Z = \frac{X-\mu}{\sigma} = \frac{X-3}{3}$$

First question: We have

$$\begin{aligned} \mathbf{P}(2 < X < 5) &= \mathbf{P}\left(-\frac{1}{3} < Z < \frac{2}{3}\right) \\ &= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) \\ &\simeq .3779 \end{aligned}$$

Simple normal computation (2)

Second question: We have

$$\begin{aligned}\mathbf{P}(X>0) &= \mathbf{P}\left(Z>-1\right) \\ &= 1-\Phi\left(-1\right) \\ &= \Phi(1) \\ &\simeq .8413\end{aligned}$$

Third question: We have $1-\mathbf{P}(-6<X-3<6)$

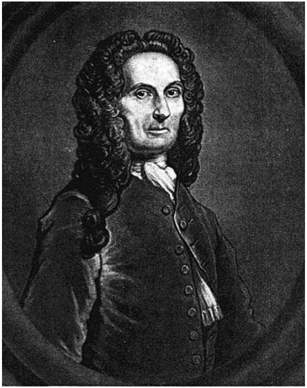
$$\begin{aligned}\mathbf{P}(|X-3|>6) &= \mathbf{P}\left(|Z|>2\right) \\ &= 1-\Phi\left(2\right)+\Phi\left(-2\right) \\ &= 2\left[1-\Phi(2)\right] \\ &\simeq .0456\end{aligned}$$

Abraham de Moivre

De Moivre $n! \sim c \left(\frac{n}{e}\right)^n \sqrt{n}$
Stirling : $c = \sqrt{2\pi}$

Some facts about de Moivre:

- Lifespan: 1667-1754, in $\simeq$ Paris, London
- Ousted from France as a protestant
  $\hookrightarrow$ in $\simeq$ 1687
- In London lived from
  - Private lessons
  - Assisting gamblers in a coffee house
- Contributions in math
  - Stirling's formula
  - First central limit theorem
  - First results on Poisson distribution



DeMoivre-Laplace theorem

Theorem 10.

Let

- $n \geq 1,\ p \in (0,1)$
- $X_n \sim \text{Bin}(n, p)$
- $a < b$

$$\frac{X_n - np}{(np(1-p))^{\frac{1}{2}}} \approx \mathcal{N}(0,1) \quad \text{as } n \to \infty$$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(a < \frac{X_n - \overbrace{np}^{E[X_n]}}{\underbrace{(np(1-p))^{1/2}}_{\sigma_{X_n}}} < b\right) = \Phi(b) - \Phi(a)$$

Empirical rule:

Accept approximation as long as $np(1-p) \geq 10$