DeMoivre-Laplace theorem

$p = 1/2$ general p

Theorem 10.

Let

- $n \geq 1,\ p \in (0,1)$
- $X_n \sim \text{Bin}(n,p)$
- $a < b$

$$\boxed{\frac{X_n - np}{(np(1-p))^{1/2}} \approx \mathcal{N}(0,1) \quad \text{as } n \to \infty}$$

where $Z \sim \mathcal{N}(0,1)$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(a < \frac{X_n - \overbrace{np}^{E[X_n]}}{\underbrace{(np(1-p))^{1/2}}_{\sigma_{X_n}}} < b\right) = \underbrace{\Phi(b)}_{= P(a < Z < b)} - \Phi(a)$$

Empirical rule:

Accept approximation as long as $np(1-p) \geq 10$

Binomial converging to normal: illustration

Rmk: distributions are not symmetric, since $p \neq \frac{1}{2}$

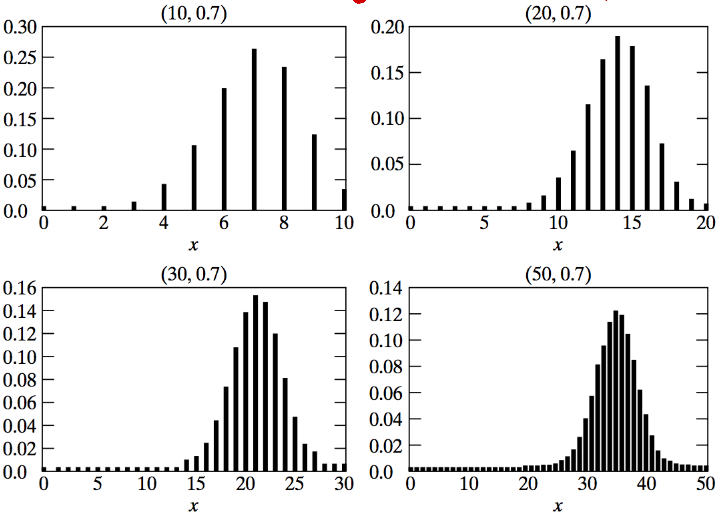

Figure: Binomial histograms for different values of (n, p)

Example: enrollment overbooking (1)

Situation:

- Ideal size of a first-year class at a particular college is 150 students.
- Data: on average, only 30% of those accepted for admission will actually attend
- College policy: approve the applications of 450 students.

Question:
Compute the probability that
more than 150 first-year students attend this college.

Model with Bin

$n = 450$ accepted
$p = .3$ prob to attend

We introduce some r.v: for each student i

$$X_i = \begin{cases} 1 & \text{if student } i \text{ attends} \\ 0 & \text{otherwise} \end{cases}$$

Then $\boxed{X_i \sim B(.3)}$ and X_i's are independent

$n = 450$ $p = .3$

Now set

X = Total number of students attending

$= \sum_{i=1}^{n} X_i$ " = # successes in Bernoulli trial "

$\sim \text{Bin}(450, .3)$

We wish to compute

$$\boxed{\mathbb{P}(X > 150) = \sum_{j=151}^{450} \binom{450}{j} .3^{j} \; .7^{450-j}}$$

long computation
approximation needed

Here $X \sim Bin(n, p)$

large (450) $\rightarrow$ 0.3, not small

$\Rightarrow$ Normal approximation is valid

empirical rule

$\boxed{\mathbb{P}(X > 150.5)}$

$$= \mathbb{P}\left(\frac{X - 450 \times .3}{(450 \times .3 \times .7)^{\frac{1}{2}}} > \overbrace{\frac{150.5 - 450 \times .3}{(450 \times .3 \times .7)^{\frac{1}{2}}}}^{= 1.59}\right)$$

De Moivre

$$\simeq \mathbb{P}(Z > 1.59)$$

$$= 1 - \phi(1.59)$$

$$= 1 - .9441 \quad \boxed{\simeq 5.59\%}$$

Example: enrollment overbooking (2)

Notation: We define

- $n = 450$, $p = .3$
- $X_i = \mathbf{1}_{(i\text{-th accepted student attends})}$, for $i = 1, \ldots, n$

Hypothesis:

- X_i i.i.d with common law $\mathcal{B}(p)$

Random variable of interest: Set

$$X = \# \text{ students that will attend}$$

Then

$$X = \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$$

Example: enrollment overbooking (3)

Normal approximation: We look for

$$\begin{aligned}\mathbf{P}(X \geq 150.5) \\ &= \mathbf{P}\left(\frac{X - 450 \times 0.3}{(450 \times 0.3 \times 0.7)^{1/2}} \geq \frac{150.5 - 450 \times 0.3}{(450 \times 0.3 \times 0.7)^{1/2}}\right)\end{aligned}$$

Therefore by DeMoivre-Laplace,

$$\mathbf{P}(X \geq 150.5) \simeq 1 - \Phi(1.59) \simeq 5.59\%$$

Outline

Exponential random variable (1)

Notation:

$$\mathcal{E}(\lambda), \text{ with } \lambda > 0$$

State space:

$$\mathbb{R}_+ = [0, \infty)$$

Density:

$$f(x) = \lambda e^{-\lambda x} \overbrace{\mathbf{1}_{\mathbb{R}_+}(x)}^{x \text{ always } \geq 0}$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{1}{\lambda}, \qquad \mathbf{Var}(X) = \frac{1}{\lambda^2}$$

Security check If f is a density, we must have

$$f \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\,dx = 1$$

For $X \sim \mathcal{E}(\lambda)$,

$$\int e^{-\alpha x}dx = \frac{-e^{-\alpha x}}{\alpha} + c$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx} = \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)\,dx$$

$$= \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$e^{-\infty} = 0$$
$$e^{0} = 1$$

$$= -e^{-\lambda x}\Big|_{0}^{\infty}$$

$$= -(0-1) \boxed{= 1} \longrightarrow$$ security check ok

$u' = 1 \quad v = -e^{-\lambda x}$

$\mathbb{E}[X]$ for $X \sim \mathcal{E}(\lambda)$

$$\boxed{\mathbb{E}[X]} = \int_{-\infty}^{\infty} x \, f(x) \, dx = \int_{-\infty}^{\infty} x \, \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+} \, dx$$

$$= \int_0^{\infty} \overset{u}{x} \, \overset{v'}{\lambda e^{-\lambda x}} \, dx \longrightarrow \text{ibp}$$

$$\overset{\text{ibp}}{=} -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} 1 \times e^{-\lambda x} \, dx$$

$$= -(0 - 0) - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^{\infty}$$

$$= -\frac{1}{\lambda}(0 - 1)$$

$$\boxed{= \frac{1}{\lambda}}$$