Example of combination (1)

Situation:

- We have a group of 5 women and 7 men
- We wish to form a committee with 2 women and 3 men

Problem:

- Find the number of possibilities

Committee example. Split the group into 5 W, 7 M. Then

Experiment 1: Pick 2 W among the 5W

possibilities: $\binom{5}{2}$

Experiment 2: Pick 3 M among the 7 M

possibilities: $\binom{7}{3}$

Total # of possible committees:

$$\binom{5}{2}\binom{7}{3} = 350$$

Example of combination (2)

Number of possibilities:

$$\binom{5}{2}\binom{7}{3} = 350$$

Example of combination (3)

Situation 2:

- We have a group of 5 women and 7 men
- We wish to form a committee with 2 women and 3 men
- 2 men refuse to serve together

Problem:

- Find the number of possibilities

Committee example with 2 grouchy guys

(i) We write

\# committees with 2 G not serving together

= # committees with 2 W, 3 M) = 350

− # committees with 2 W, 3 M and 2 G serving together) = Q

committees with 2W, 3M and 2 G serving together $) \equiv Q$

(ii) In order to compute Q, we divide the 12 persons into 5W, 2G, 5M. Then

Experiment 1: Pick 2W among 5 $\rightarrow \binom{5}{2}$

Experiment 2: Pick 2G among 2 $\rightarrow \binom{2}{2}$

Experiment 3: Pick 1M among 5 $\rightarrow \binom{5}{1}$

Thus $\boxed{Q = \binom{5}{2}\binom{2}{2}\binom{5}{1} = 50}$

Conclusion

Committees with 2W, 3M, 2G not serving together

$= 350 - 50$

$= 300$

Example of combination (4)

New number of possibilities:

$$\binom{5}{2}\left\{\binom{7}{3}-\binom{2}{2}\binom{5}{1}\right\}=300$$

Binomial theorem

Theorem 3.

Let

- $x_1, x_2 \in \mathbb{R}$
- $n \geq 1$

Then

$$(x_1 + x_2)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k \, x_2^{n-k}$$

Combinatorial proof

First expansion:

$$(x_1 + x_2)^n = \sum_{(i_1,\ldots,i_n)\in\{1,2\}^n} x_{i_1}\, x_{i_2} \cdots x_{i_n}$$

Definition of a family of sets:

$$A_k = \left\{(i_1, \ldots, i_n) \in \{1,2\}^n;\ \text{there are } k\ j\text{'s such that } i_j = 1\right\}.$$

New expansion: we have (convention: $|A_k| \equiv \text{Card}(A_k)$)

$$\begin{aligned}(x_1 + x_2)^n &= \sum_{k=0}^{n} |A_k|\, x_1^k\, x_2^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x_1^k\, x_2^{n-k}\end{aligned}$$

Application of the binomial theorem

Proposition 4.

Let

- A a set with $|A| = n$ Cardinality of A
- $\mathcal{P}_n \equiv$ collection of all subsets of A

Then

$$|\mathcal{P}_n| = 2^n$$

Example with $n = 2$. Take

- $A = \{ a , b \} \Rightarrow |A| = 2$

Then

- $\mathcal{P}_n = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \}$

Thus

$$\boxed{|\mathcal{P}_n| = 4 = 2^2}$$

→ compatible with Prop 4

Proof of Prop 4

$$|P_n| = \sum_{k=0}^{n} \#\{\text{subsets with } k \text{ elements}\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \, 1^{k} \, 1^{n-k}$$

binomial

$$= (1+1)^n$$

$$= 2^n$$

Proof

Decomposition of $|\mathcal{P}_n|$: Write

$$
\begin{aligned}
|\mathcal{P}_n| &= \sum_{k=0}^{n} |\text{Subsets of } A \text{ with } k \text{ elements}| \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}
$$

Application of the binomial theorem:

$$
\begin{aligned}
|\mathcal{P}_n| &= (1+1)^n \\
&= 2^n
\end{aligned}
$$

Outline

Multinomial coefficients

Divisions of n objects into r groups with size $n_1, \ldots, n_r$: We have

- n objects and r groups
- We want n_j objects in group j and $\sum_{j=1}^{r} n_j = n$

Notation: Set

$$\binom{n}{n_1, \ldots, n_r} = \frac{n!}{\prod_{j=1}^{r} (n_j!)}$$

Counting: We have

$$\# \text{ Divisions of } n \text{ objects into } r \text{ groups with size } n_1, \ldots, n_r = \binom{n}{n_1, \ldots, n_r}$$

Proof of counting

Number of choices for the ith group:

$$\binom{n - \sum_{j=1}^{i-1} n_j}{n_i}$$

Number of divisions: We have

$$\begin{aligned}
&\# \text{ Divisions of } n \text{ objects into } r \text{ groups with size } n_1, \ldots, n_r \\
&= \prod_{i=1}^{r} \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} \\
&= \binom{n}{n_1, \ldots, n_r}
\end{aligned}$$

Example of multinomial coefficient (1)

Situation: Police department with 10 officers and

- 5 have to patrol the streets
- 2 are permanently working at the station
- 3 are on reserve at the station

Problem:
How many divisions do we get?

Police department example

\# of possible divisions

$$= \binom{10}{5,2,3}$$

$$= \boxed{\frac{10!}{5!\ 2!\ 3!}}$$

with $r = 3$

$n_1 = 5$

$n_2 = 2$

$n_3 = 3$

Example of multinomial coefficient (2)

Answer:

$$\frac{10!}{5!\,2!\,3!} = 2520$$

Tournament example

Situation: Tournament with $n = 2^m$ players
$\hookrightarrow$ How many outcomes?

Particular case:
Take $m = 3$, thus $n = 8$

Number of rounds: 3

Tournament example (2)

Counting number of outcomes for the first round:

$$\overbrace{\binom{8}{2,2,2,2}}^{\#\text{ pairings with order}} \; \overbrace{\frac{1}{4!}}^{\text{No ordering}} \; \overbrace{2^4}^{\text{Possible outcomes}} = \frac{8!}{4!}$$

Counting number of outcomes for second and third round:

$$\frac{4!}{2!} \quad \text{and} \quad \frac{2!}{1!}$$

Conclusion:

$$\frac{8!}{4!}\,\frac{4!}{2!}\,\frac{2!}{1!} = 8! = 40,320 \text{ possible outcomes}$$