Exponential random variable (2)

Use: Waiting time between

- 2 customer arrivals in a shop on a typical afternoon
- Bus arrivals at a bus stop
- Two jobs on a server from 12am to 6am

queuing theory

Empirical rule:

Number of arrivals given by a Poisson random variable

$$\Longrightarrow$$

Inter arrivals given by exponential random variables

Tail probability: If $X \sim \mathcal{E}(\lambda)$, then for $x \geq 0$ we have

$$\mathbf{P}(X > x) = \int_x^\infty \lambda\, e^{-\lambda z}\, dz = e^{-\lambda x}$$

Computation of the tail for $X \sim \mathcal{E}(\lambda)$

For $x \geq 0$,

$$\boxed{\mathbb{P}(X > x)} = \int_x^\infty \lambda e^{-\lambda z}\, dz$$

$$= -e^{-\lambda z} \Big|_x^\infty$$

$$= -\left(0 - e^{-\lambda x}\right)$$

$$\Rightarrow \mathbb{P}(X > x) \boxed{= e^{-\lambda x}}$$

Graphing an exponential law

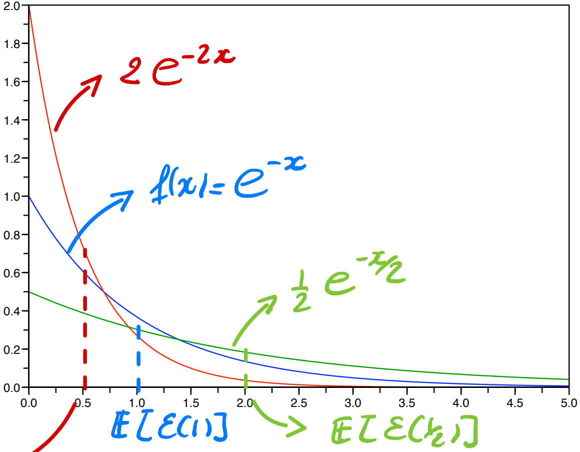

Figure: $\mathcal{E}(1)$, $\mathcal{E}(2)$, $\mathcal{E}(1/2)$. x-axis: x. y-axis: $f(x)$

Memoryless property

Proposition 11.

Let

- X be continuous random variable

Then X satisfies the memoryless property

$$\mathbf{P}\left(X > s + t \middle| X > t\right) = \mathbf{P}\left(X > s\right)$$

if and only if there exists $\lambda > 0$ such that

$$X \sim \mathcal{E}(\lambda)$$

Memoryless property for $\mathcal{E}(\lambda)$.

$s,t \geq 0$

Let $X \sim \mathcal{E}(\lambda)$. Then

$$\boxed{\mathbb{P}(\overbrace{X > s+t}^{A} \mid \overbrace{X > t}^{B})}$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}((X > s+t) \cap (X > t))}{\mathbb{P}(X > t)}$$

$$= \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > t)}$$

$$\overset{\text{Tail}}{=} \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} \overset{\text{Tail}}{=} \boxed{\mathbb{P}(X > s)}$$

Interpretation of memoryless.

If X models a lifetime. Then

$$\underbrace{\mathbb{P}(X > t+s \mid X > t)}_{\text{probab to be alive after } s \text{ instants after } t \text{ given that we are alive at time } t} = \underbrace{\mathbb{P}(X > s)}_{\text{probab to be alive } s \text{ instants after birth}}$$

Conclusion : $\boxed{\text{X models systems with no aging}}$

Proof of $\Longrightarrow$ (1)

Functional equation: Set

$$\bar{F}(x) = \mathbf{P}(X > x)$$

Then if X is memoryless, $\bar{F}$ satisfies

$$g(s+t) = g(s)\, g(t) \tag{3}$$

Value of g on rationals: If g satisfies (3), then

$$g\left(\frac{1}{n}\right) = (g(1))^{1/n}, \qquad g\left(\frac{m}{n}\right) = (g(1))^{m/n}$$

Proof of $\Longrightarrow$ (2)

Expression for $g(1)$:
We have $g(1) = [g(1/2)]^2 \geq 0$. Thus there exists $\lambda \in \mathbb{R}$ such that

$$g(1) = e^{-\lambda}$$

Value of g on rationals (2): We have found that for $x \in \mathbb{Q}_+$,

$$g(x) = e^{-\lambda x}$$

Conclusion: By continuity of g, for all $x \in \mathbb{R}_+$ we have

$$g(x) = e^{-\lambda x}$$

Example: car battery (1)

Situation:

- Number of miles that a car can run before its battery wears out is exponentially distributed
- Average value of 10k miles
- We have already run 3k miles with the battery
- We wish to take a 5k trip

Question: Probability to complete the trip without having to replace the car battery?

Model : $X \sim \mathcal{E}(\lambda)$ and $\mathbb{E}[X] = 10$

Since $\mathbb{E}[X] = \frac{1}{\lambda}$ we have $\lambda = \frac{1}{10}$

We wish to compute

$$\boxed{\mathbb{P}(X > 3 + 5 \mid X > 3)}$$

No aging

$$= \mathbb{P}(X > 5)$$

Tail

$$= e^{-5\lambda}$$

$$= e^{-5/10} = e^{-\frac{1}{2}} \quad \boxed{\simeq 60\%}$$

Example: car battery (1)

Model:

- $X = \#$ miles before battery wears out
- $X \sim \mathcal{E}(\lambda)$
- $\lambda = \frac{1}{\mathbf{E}[X]} = \frac{1}{10}$
- We wish to compute $\mathbf{P}(X > 3 + 5 | X > 3)$

Computation:

$$\mathbf{P}\left(X > 3 + 5 | X > 3\right) = \mathbf{P}\left(X > 5\right) = e^{-\frac{1}{2}} \simeq 0.604$$

Hazard rate function (1)

Definition 12.

Let

- X positive continuous random variable
- Density f, cdf F
- $\bar{F} = 1 - F$ $\qquad \bar{F}(x) = \mathbb{P}(X > x)$

Then the hazard rate function is given by

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}$$

Hazard rate function (2)

Interpretation: λ is a failure rate, i.e

$$\mathbf{P}\left(X \in [t, t + dt) \middle| X > t\right) \simeq \lambda(t)\, dt$$

Exponential case: If $X \sim \mathcal{E}(\lambda)$, we have

$$\lambda(t) = \lambda$$

Hazard rate function (3)

Cdf from λ: from the relation

$$\lambda(t) = \frac{F'(t)}{1 - F(t)},$$

we get

$$F(t) = 1 - \exp\left(-\int_0^t \lambda(s)\, ds\right)$$

Survival probability from λ: For $a, b \geq 0$,

$$\mathbf{P}\left(X > a + b \mid X > a\right) = \exp\left(-\int_a^{a+b} \lambda(s)\, ds\right)$$

Example: smokers survival (1)

Data:

- Death rate of smokers = twice death rate of non smokers
- Consider 2 40-years old persons, 1 S and 1 N
- We wish to compare their probability to survive until 50

Model: Let

$$\lambda_n = \text{ hazard rate for N}, \qquad \lambda_s = \text{ hazard rate for S}$$

Then

$$\lambda_s = 2\,\lambda_n$$

Example: smokers survival (2)

Compute:

$$
\begin{aligned}
\mathbf{P}(S > 50 | S > 40) &= \exp\left(-\int_{40}^{50} \lambda_s(r)\, dr\right) \\
&= \exp\left(-2\int_{40}^{50} \lambda_n(r)\, dr\right) \\
&= [\mathbf{P}(N > 50 | N > 40)]^2
\end{aligned}
$$

$$= \left(\exp\left(-\int_{40}^{50} \lambda_n(r)\, dr\right)\right)^2$$