Outline

Joint cdf

notice which is valid for any couple (X,Y)

Definition 1.

Let

- X, Y random variables
- $a, b \in \mathbb{R}$

The joint cdf describes the joint distribution of (X, Y):

$$F(a, b) = \mathbf{P}\,\underbrace{(X \le a,\ Y \le b)}$$

stands for $(X \le a) \cap (Y \le 6)$

Values of interest in terms of the cdf

Proposition 2.

Let

- X, Y random variables
- F the joint cdf of X, Y

Then the marginals cdf's of X and Y are given by

$= \lim_{b \to \infty} F(a, b)$

$$F_X(a) = F(a, \infty), \qquad F_Y(b) = F(\infty, b)$$

$= \lim_{a \to \infty} F(a, b)$

We also have

$$\mathbf{P}(a_1 < X \le a_2,\ b_1 < Y \le b_2) = F(a_2, b_2) - F(a_2, b_1) - F(a_1, b_2) + F(a_1, b_1)$$

Discrete case: joint pmf

Definition 3.

Consider the following situation:

- X, Y discrete random variables
- X takes values in E_1, Y takes values in E_2
- Both E_1 and E_2 are countable
- $x \in E_1$ and $y \in E_2$

The joint pmf p describes the joint distribution of (X, Y):

$$p(x, y) = \mathbf{P}\left(X = x,\ Y = y\right)$$

Values of interest in terms of the pmf

Proposition 4.

Let

- X, Y random variables
- p the joint pmf of X, Y

sum over b

Then the marginals pmf's of X and Y are given by

b frozen

a frozen

$$p_X(a) = \sum_{b \in E_2} p(a, b), \qquad p_Y(b) = \sum_{a \in E_1} p(a, b)$$

If $a_1 < a_2$ and $b_1 < b_2$, we also have

$$\mathbf{P}\left(a_1 < X \leq a_2,\, b_1 < Y \leq b_2\right) = \sum_{a_1 < i_1 \leq a_2,\, b_1 < i_2 \leq b_2} p\left(i_1, i_2\right)$$

Example: tossing 3 coins (1)

Sample space :
$S = \{h, t\}^3$

Experiment:
Tossing a coin 3 times

Events: We consider

$$A = \text{"At most one Head"}$$
$$B = \text{"At least one Head and one Tail"}$$

Random variables: Set

$$X_1 = \mathbf{1}_A, \qquad X_2 = \mathbf{1}_B, \qquad X = (X_1, X_2)$$

$= \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

Question: pmf for X ?

s	$X(s)$	s	$X(s)$
(t,t,t)	$(1,0)$	(h,t,t)	$(1,1)$
(t,t,h)	$(1,1)$	(h,t,h)	$(0,1)$
(t,h,t)	$(1,1)$	(h,h,t)	$(0,1)$
(t,h,h)	$(0,1)$	(h,h,h)	$(0,0)$

A = at most 1 h $\quad X_1 = \mathbb{1}_A$

B = at least 1 h + 1 t $\quad X_2 = \mathbb{1}_B$

s	$X(s)$	s	$X(s)$
(t,t,t)	$(1,0)$	(h,t,t)	$(1,1)$
(t,t,h)	$(1,1)$	(h,t,h)	$(0,1)$
(t,h,t)	$(1,1)$	(h,h,t)	$(0,1)$
(t,h,h)	$(0,1)$	(h,h,h)	$(0,0)$

Pmf: Given by the 4 values

$$p(0,0) = \mathbb{P}(X = (0,0)) = \frac{1}{8}$$

$$p(0,1) = \mathbb{P}(X = (0,1)) = \frac{3}{8}$$

$$p(1,0) = \mathbb{P}(X = (1,0)) = \frac{1}{8}$$

$$p(1,1) = \mathbb{P}(X = (1,1)) = \frac{3}{8}$$

Summarizing in a table

$X_1 \setminus X_2$	0	1	Marg X_1
0	$1/8$	$3/8$	$1/2$
1	$1/8$	$3/8$	$1/2$
Marg X_2	$1/4$	$3/4$	1

$X_1 \sim B(\tfrac{1}{2})$

$X_2 \sim B(3/4)$

$p(0,0) = \mathbb{P}(X = (0,0)) = \frac{1}{8}$

$p(0,1) = \mathbb{P}(X = (0,1)) = \frac{3}{8}$

$p(1,0) = \mathbb{P}(X = (1,0)) = \frac{1}{8}$

$p(1,1) = \mathbb{P}(X = (1,1)) = \frac{3}{8}$

Marginals with formula

$$\boxed{\mathbb{P}(X_1 = 0)} = \sum_{i=0}^{1} \mathbb{P}(X_1 = 0, X_2 = i)$$

$$= \mathbb{P}(X_1 = 0, X_2 = 0) + \mathbb{P}(X_1 = 0, X_2 = 1)$$

$$= 1/8 + 3/8 \quad \boxed{= 1/2}$$

$$\boxed{\mathbb{P}(X_1 = 1)} = \sum_{i=0}^{1} \mathbb{P}(X_1 = 1, X_2 = i)$$

$$= \frac{1}{8} + \frac{3}{8}$$

$$\boxed{= \frac{1}{2}}$$

$p(0,0) = \mathbb{P}(X = (0,0)) = \frac{1}{8}$

$p(0,1) = \mathbb{P}(X = (0,1)) = \frac{3}{8}$

$p(1,0) = \mathbb{P}(X = (1,0)) = \frac{1}{8}$

$p(1,1) = \mathbb{P}(X = (1,1)) = \frac{3}{8}$

Example: tossing 3 coins (2)

Model: We take

- $S = \{h, t\}^3$
- $\mathbf{P}(\{s\}) = \frac{1}{8}$ for all $s \in S$

Description of $X = (X_1, X_2)$:

s	$X(s)$	s	$X(s)$
(t, t, t)	$(1, 0)$	(h, t, t)	$(1, 1)$
(t, t, h)	$(1, 1)$	(h, t, h)	$(0, 1)$
(t, h, t)	$(1, 1)$	(h, h, t)	$(0, 1)$
(t, h, h)	$(0, 1)$	(h, h, h)	$(0, 0)$

Example: tossing 3 coins (3)

Joint pmf for X:

$$\mathbf{P}(X=(0,0)) = \frac{1}{8}, \quad \mathbf{P}(X=(0,1)) = \frac{3}{8}$$
$$\mathbf{P}(X=(1,0)) = \frac{1}{8}, \quad \mathbf{P}(X=(1,1)) = \frac{3}{8}$$

Marginal pmf for X_1:

$$\begin{aligned}
\mathbf{P}(X_1=0) &= \sum_{i=0}^{1} \mathbf{P}(X=(0,i)) \\
&= \mathbf{P}(X=(0,0)) + \mathbf{P}(X=(0,1)) \\
&= \frac{1}{8} + \frac{3}{8} = \frac{1}{2} \\
\mathbf{P}(X_1=1) &= \frac{1}{2}
\end{aligned}$$

Example: tossing 3 coins (4)

Marginal pmf for X_2:

$$\mathbf{P}(X_2 = 0) = \frac{1}{4}, \quad \mathbf{P}(X_2 = 1) = \frac{3}{4}$$

Remark:

We have $X_1 \sim \mathcal{B}(1/2)$ and $X_2 \sim \mathcal{B}(3/4)$

Summary in a table:

$X_1 \backslash X_2$	0	1	Marg. X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg. X_2	1/4	3/4	1

Continuous case: joint density

Again discrete $\to$ continuous boils down to
$\Sigma \longrightarrow \int$

Definition 5.

Consider the following situation:

- X, Y continuous real valued random variables

The random vector (X, Y) is said to be jointly continuous iff for "all" subsets $C \subset \mathbb{R}^2$ we have

$$\mathbf{P}((X, Y) \in C) = \int\int_{(x,y)\in C} f(x, y)\, dxdy$$

see Calc 3

Values of interest in terms of the density

Proposition 6.

Let

- X, Y random variables
- f the joint density of X, Y

Then the marginals densities of X and Y are given by

freeze x   integrate over y

$$f_X(x) = \int_{\mathbb{R}} f(x, y)\, dy \qquad f_Y(y) = \int_{\mathbb{R}} f(x, y)\, dx$$

If $a_1 < a_2$ and $b_1 < b_2$, we also have

$$\mathbf{P}\left(a_1 < X \le a_2,\ b_1 < Y \le b_2\right) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f(x, y)\, dxdy$$

Simple example of bivariate density (1)

Density: Let (X, Y) be a random vector with density

$$2e^{-x}e^{-2y}\,\mathbf{1}_{(0,\infty)}(x)\,\mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

$$\mathbf{P}(X < Y)$$

Security check for $f(x,y)$. If f is a density, we should have

(i) $f(x,y) \geq 0$

(ii) $\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$

Here

$$f(x,y) = 2e^{-x} e^{-2y} \mathbb{1}_{\mathbb{R}_+}(x) \mathbb{1}_{\mathbb{R}_+}(y)$$

It is obvious that $\boxed{f(x,y) \geq 0}$

Moreover

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy}$$

$$= \int_0^{\infty}\int_0^{\infty} 2e^{-x}\, e^{-2y}\, dx\, dy$$

$$= \int_0^{\infty} e^{-x}\, dx \times \int_0^{\infty} 2e^{-2y}\, dy$$

$$= -e^{-x}\Big|_0^{\infty} \qquad -e^{-2y}\Big|_0^{\infty}$$

$$= 1 \times 1$$

$$\boxed{= 1}$$

$\longrightarrow$ security check ok!