Simple example of bivariate density (1)

Density: Let (X, Y) be a random vector with density

$$2e^{-x}e^{-2y}\,\mathbf{1}_{(0,\infty)}(x)\,\mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

$$\mathbf{P}(X < Y)$$

$f(x,y) = 2e^{-x}e^{-2y}\mathbb{1}_{\mathbb{R}_+}(x)\mathbb{1}_{\mathbb{R}_+}(y)$

Computation of $\mathbb{P}(X<Y)$. We have

$$\mathbb{P}(X \leq Y) = \int_{0\leq x<y<\infty} f(x,y)\,dx\,dy$$

$$= \int_0^\infty \left(\int_0^y 2e^{-x}e^{-2y}\,dx\right)dy$$

$$-e^{-x}\Big|_0^y = 1-e^{-y}$$

$$= \int_0^\infty 2e^{-2y}\left(\int_0^y e^{-x}\,dx\right)dy$$

$$= \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy$$

Summary

$$P(X<Y) = \int_0^\infty 2e^{-2y}(1-e^{-y})\,dy$$

$$= \int_0^\infty 2e^{-2y}\,dy - 2\int_0^\infty e^{-3y}\,dy$$

$$= -e^{-2y}\Big|_0^\infty + \frac{2}{3}\,e^{-3y}\Big|_0^\infty$$

$$= 1 - \frac{2}{3}$$

$$\Rightarrow \boxed{P(X<Y) = \frac{1}{3}}$$

Simple example of bivariate density (2)

Computation: We have

$$\begin{aligned}
\mathbf{P}(X < Y) &= 2\int_{0<x<y<\infty} e^{-x} e^{-2y}\, dxdy \\
&= 2\int_0^\infty dy\, e^{-2y} \int_0^y e^{-x}\, dx \\
&= 2\int_0^\infty e^{-2y}(1 - e^{-y})\, dy \\
&= \frac{1}{3}
\end{aligned}$$

Change of variable: general result

Theorem 7.

Let

- $X = (X_1, X_2)$ continuous random variable
- Density: f_X
- g diffeomorphism of $\mathbb{R}^2$
- $Y = g(X)$

invertible + additional condition

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Then Y has a density f_Y given by

$= \text{"}\frac{dy}{dx}\text{"}$

$$f_Y(y) = f_X\left(g^{-1}(y)\right) J(y)\, \underbrace{\mathbf{1}_{\{y=g(x) \text{ for some } x\}}}_{= 1 \text{ most of the time}}$$

Change of variable in the plane (1)

Density: Let (X, Y) be a random vector with density

$$e^{-(x+y)} \mathbf{1}_{(0,\infty)}(x) \mathbf{1}_{(0,\infty)}(y)$$

Question:
Compute the density of the r.v $Z = \frac{X}{Y}$

Strategy: (i) Get the density of
$(Z, W) = \left(\frac{X}{Y}, Y\right) \Rightarrow g(x,y) = (z,w) = \left(\frac{x}{y}, y\right)$
(ii) Get f_Z as the marginal density for (Z, W)

$f_{X,Y}(x,y) = e^{-(x+y)} \mathbb{1}_{\mathbb{R}_+}(x) \mathbb{1}_{\mathbb{R}_+}(y)$

Step (i): Density for $(Z,W) = \left(\frac{X}{Y}, Y\right)$. We have

if $z \geq 0, \omega \geq 0$

$$f_{Z,W}(z,\omega) = f_{X,Y}\left(g^{-1}(z,\omega)\right) J(z,\omega) \times 1$$

Moreover

$$z = \frac{x}{y}, \quad \omega = y$$

$$\Rightarrow x = yz = \omega z, \quad y = \omega$$

$$\Rightarrow g^{-1}(z,\omega) = (\omega z, \omega)$$

$$\Rightarrow \boxed{f_{X,Y}\left(g^{-1}(z,\omega)\right)} = e^{-(\omega z + \omega)} \boxed{= e^{-\omega(z+1)}}$$

if $z, \omega \geq 0$

Summary

$$f_{z,w}(z,\omega) = \overbrace{f_{x,y}\left(g^{-1}(z,\omega)\right)}^{= e^{-\omega(z+1)}} J(z,\omega)$$

Jacobian :

$$x = z\omega$$
$$y = \omega$$

$$\Rightarrow J(z,\omega) = \left| \det \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial \omega} & \frac{\partial y}{\partial \omega} \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} \omega & 0 \\ z & 1 \end{pmatrix} \right| = \omega$$

$$\Rightarrow \boxed{f_{z,w}(z,\omega) = e^{-\omega(z+1)} \, \omega}$$

Step (ii) : Marginal f_Z

$$\boxed{f_{Z,W}(z,w)} = e^{-w(z+1)} \, w$$

$$f_Z(z) = \int_0^\infty f_{Z,W}(z,w) \, dw$$

$$= \int_0^\infty w \, e^{-(z+1)w} \, dw$$

$$\overset{\text{ibp}}{=} \dots \quad \boxed{\frac{1}{(z+1)^2} \quad \text{if } z \geq 0}$$

Change of variable in the plane (2)

Characterization through expectations: Let $\varphi \in \mathcal{C}_b(\mathbb{R})$. Then

$$\mathbf{E}\left[\varphi(Z)\right] = \int_0^\infty \int_0^\infty \varphi\left(\frac{x}{y}\right) e^{-(x+y)}\, dxdy$$

Change of variable: Set

$$z = \frac{x}{y}, \quad w = y \quad \Longleftrightarrow \quad x = z\, w, \quad y = w$$

Jacobian:

$$J = w$$

Change of variable in the plane (3)

Computing $\mathbf{E}[\varphi(Z)]$:

$$\begin{aligned}
\mathbf{E}\left[\varphi(Z)\right] &= \int_0^\infty \int_0^\infty \varphi(z)\, we^{-w(z+1)}\, dwdz \\
&= \int_0^\infty dz\, \varphi(z) \int_0^\infty w\, e^{-w(z+1)}\, dw \\
&= \int_0^\infty \varphi(z)\, \frac{1}{(1+z)^2}\, dz
\end{aligned}$$

Density of Z:

$$\frac{1}{(1+z)^2}\, \mathbf{1}_{(0,\infty)}(z)$$