Joint cdf in higher dimensions

Definition 8.

Let

- $X_1, \ldots, X_n$ random variables $\rightarrow X = (X_1, .., X_n)$
- $a_1, \ldots, a_n \in \mathbb{R}$

The following joint cdf describes
the joint distribution of $(X_1, \ldots, X_n)$:

$$F(a_1, \ldots, a_n) = \mathbf{P}\left(X_1 \le a_1, \ldots, X_n \le a_n\right)$$

Joint density in higher dimensions

Definition 9.

Consider the following situation:

- $X_1, \ldots, X_n$ real valued random variables

The random vector $(X_1, \ldots, X_n)$ is said to be jointly continuous iff for "all" subsets $C \subset \mathbb{R}^n$ we have

$$\mathbf{P}\left((X_1, \ldots, X_n) \in C\right) = \int_{(x_1, \ldots, x_n) \in C} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

Outline

Recall $A \perp\!\!\!\perp B$ if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

For r.v's, we will relate the notion $X \perp\!\!\!\perp Y$ to indep. of events

Definition of independence

Definition 10.

Let

- X, Y random variables

X and Y are said to be independent if for "all" $C, D \subset \mathbb{R}$ we have

$$\mathbf{P}\left(\overbrace{X \in C}^{A} \overset{\cap}{,} \overbrace{Y \in D}^{B}\right) = \mathbf{P}\left(X \in C\right) \mathbf{P}\left(Y \in D\right)$$

otherwise stated $(X \in C) \perp\!\!\!\perp (Y \in D)$
for all C, D

Characterizations of independence

Proposition 11.

Let X, Y random variables.
Then X and Y are independent in the following cases

1. If the joint cdf F satisfies
$$F(a, b) = F_X(a)\, F_Y(b), \quad \text{for all } a, b \in \mathbb{R}$$
2. If X, Y are discrete and the joint pmf satisfies
$$p(x, y) = p_X(x)\, p_Y(y), \quad \text{for all } (x, y) \in E_1 \times E_2$$
3. If X, Y are jointly cont. and the joint density satisfies
$$f(x, y) = f_X(x)\, f_Y(y), \quad \text{for all } (x, y) \in \mathbb{R}^2$$

Example ctd: tossing 3 coins (1)

Experiment:
Tossing a coin 3 times

Events: We consider

$$A = \text{"At most one Head"}$$
$$B = \text{"At least one Head and one Tail"}$$

Random variables: Set

$$X_1 = \mathbf{1}_A, \qquad X_2 = \mathbf{1}_B, \qquad X = (X_1, X_2)$$

Question: Do we have $X_1 \perp\!\!\!\perp X_2$?

Strategy In order to check $X_1 \perp\!\!\!\perp X_2$,

(i) We need to check

$$\mathbb{P}(X_1 = i, X_2 = j) = \mathbb{P}(X_1 = i)\mathbb{P}(X_2 = j)$$

$$\forall\ i,j \in \{0,1\}$$ $\longrightarrow$ 4 identities

(ii) It might be easier to read those identities on the table

Table for the distribution of (X_1, X_2)

$X_1 \backslash X_2$	0	1	Marg X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg X_2	1/4	3/4	

Checking ⫫ with the table

$X_1 \setminus X_2$	0	1	Marg X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg X_2	1/4	3/4	

- $\frac{1}{8} = \mathbb{P}(X_1=0, X_2=0) = \mathbb{P}(X_1=0)\,\mathbb{P}(X_2=0) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$
- $\frac{3}{8} = \mathbb{P}(X_1=0, X_2=1) = \mathbb{P}(X_1=0)\,\mathbb{P}(X_2=1) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{8}$
- $\frac{1}{8} = \mathbb{P}(X_1=1, X_2=0) = \mathbb{P}(X_1=1)\,\mathbb{P}(X_2=0) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$
- $\frac{3}{8} = \mathbb{P}(X_1=1, X_2=1) = \mathbb{P}(X_1=1)\,\mathbb{P}(X_2=1) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$

Conclusion: $\boxed{X_1 \perp\!\!\!\perp X_2}$ → due to $A \perp\!\!\!\perp B$ for $n=3$ toss

Example ctd: tossing 3 coins (2)

We have seen:

$X_1 \backslash X_2$	0	1	Marg. X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg. X_2	1/4	3/4	1

Checking independence: With the help of the table, one can see that

$$\mathbf{P}(X = (i,j)) = \mathbf{P}(X_1 = i)\,\mathbf{P}(X_2 = j), \quad \text{for all} \quad i,j \in \{0,1\}$$

Therefore $X_1 \perp\!\!\!\perp X_2$.

Remark: The relation $X_1 \perp\!\!\!\perp X_2$ is due to the fact that $A \perp\!\!\!\perp B$.
$\hookrightarrow$ cf. Conditional probability, Section 4.

Example: Romeo and Juliet (1)

Situation:

- Romeo and Juliet decide to meet on the main square of Verona
- They arrive at independent times between 12pm and 1pm
- Rule: the first to arrive leaves after 10mn

Question:

Compute the probability that Romeo meets Juliet

Model

Set X = arrival time for R

Y = arrival time for J

Hyp 1: $X, Y \sim \mathcal{U}([12pm, 1pm])$

↪ shift to $X, Y \sim \mathcal{U}([0,1])$

Hyp 2: $X \perp\!\!\!\perp Y$

$\Rightarrow$ Joint $\boxed{f(x,y)} = \mathbb{1}_{[0,1]}(x)\ \mathbb{1}_{[0,1]}(y)$

$\boxed{= \mathbb{1}_{[0,1]^2}(x,y)}$