Romeo & Juliet example

$X =$ arrival time for R $\sim \mathcal{U}([0,1])$

$Y =$ arrival time for J $\sim \mathcal{U}([0,1])$

Hyp $X \perp\!\!\!\perp Y$

$\Rightarrow$ we have

$$f_X(x) = \mathbb{1}_{[0,1]}(x)$$

$$f_Y(y) = \mathbb{1}_{[0,1]}(y)$$

$$\boxed{f_{(X,Y)}(x,y) \overset{\perp\!\!\!\perp}{=} f_X(x)\, f_Y(y) = \mathbb{1}_{[0,1]^2}(x,y)}$$

<u>Rule</u> : R & J will wait at most $\frac{1}{6}$ h

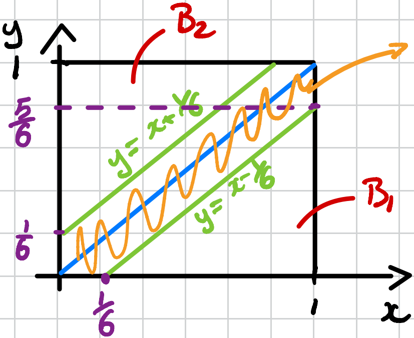
$\Rightarrow \boxed{\mathbb{P}(\text{R \& J meet})}$

$= \mathbb{P}\left(|X-Y| \leq \frac{1}{6} \right)$

$\overset{\mathcal{U}([0,1]^2)}{=}$ Area (region $\{ x,y \in [0,1]^2 : |x-y| \leq \frac{1}{6} \}$)

$\boxed{= \quad A}$

Picture for A



$A = |\{(x,y) \in [0,1]^2 ; |x-y| \leq \frac{1}{6}\}|$

We have $A = 1 - |B_1| - |B_2| = 1 - 2|B_1|$

$$\Rightarrow A = 1 - 2 \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{2} = 1 - \left(\frac{5}{6}\right)^2$$

$$\Rightarrow \boxed{\mathbb{P}(R \,\&\, J \text{ meet}) = 1 - \left(\frac{5}{6}\right)^2 \simeq 30.5\%}$$

Example: Romeo and Juliet (2)

Model:

- X = Arrival time for Romeo
- Y = Arrival time for Juliet
- Renormalize everything on $[0,1]$
- Hypothesis: $X \perp\!\!\!\perp Y$ and $X, Y \sim \mathcal{U}([0,1])$

Joint density: The joint density for (X, Y) is

$$f(x,y) = \mathbf{1}_{[0,1]^2}(x,y) = \mathbf{1}_{[0,1]}(x)\,\mathbf{1}_{[0,1]}(y)$$

Example: Romeo and Juliet (3)

Aim: Compute

$$\mathbf{P}\left(|Y - X| < \frac{1}{6}\right)$$

Complementary: Geometrically one can see that

$$\mathbf{P}\left(|Y - X| \geq \frac{1}{6}\right) = \left(\frac{5}{6}\right)^2$$

Conclusion:

$$\mathbf{P}\left(|Y - X| < \frac{1}{6}\right) = 1 - \left(\frac{5}{6}\right)^2 \simeq 30.5\%$$

Characterizations of independence

Proposition 12.

Let X, Y random variables.
Then X and Y are independent in the following cases

1. If X, Y are discrete and there exist h, g such that
$$p(x, y) = h(x)\, g(y), \quad \text{for all } (x, y) \in E_1 \times E_2$$
2. If X, Y are jointly cont. and there exist h, g such that
$$f(x, y) = h(x)\, g(y), \quad \text{for all } (x, y) \in \mathbb{R}^2$$

Example of independence (1)

Example 1: If (X, Y) have joint density

$$6e^{-(2x+3y)} \mathbf{1}_{(0,\infty)^2}(x, y),$$

then $X \perp\!\!\!\perp Y$.

Exponential example. We have

$$f(x,y) = 6e^{-(2x+3y)} \mathbb{1}_{(0,\infty)^2}(x,y)$$

$$= \underbrace{6e^{-2x} \mathbb{1}_{(0,\infty)}(x)}_{h(x)} \times \underbrace{e^{-3y} \mathbb{1}_{(0,\infty)}(y)}_{g(y)}$$

$\overset{\text{Prop 12}}{\Longrightarrow}$ $\boxed{X \perp\!\!\!\perp Y}$

Example of independence (2)

Recall joint density:

$$6e^{-(2x+3y)}\,\mathbf{1}_{(0,\infty)^2}(x,y)$$

Decomposition of the density:

$$f(x,y) = h(x)\,g(y),$$

with

$$h(x) = 6e^{-2x}\,\mathbf{1}_{(0,\infty)}(x), \qquad g(y) = e^{-3y}\,\mathbf{1}_{(0,\infty)}(y)$$

Conclusion:

$$X \perp\!\!\!\perp Y$$

Example of non independence (1)

Example 2: If (X, Y) have joint density

$$24xy\, \mathbf{1}_{(0,\infty)^2}(x, y) \mathbf{1}_{(0<x+y<1)},$$

then X, Y are not independent

Example of non $\perp\!\!\!\perp$ We have

$$f(x,y) = 24xy\, \mathbb{1}_{(0,\infty)^2}(x,y)\, \mathbb{1}_{\{x+y \leq 1\}}$$

$$= \underbrace{24x\, \mathbb{1}_{(0,\infty)}(x)}_{h(x)} \times \underbrace{y\, \mathbb{1}_{(0,\infty)}(y)}_{g(y)} \times \mathbb{1}_{\{x+y \leq 1\}}$$

$\Rightarrow$ no product decomposition

⚠ Not enough to prove $X \not\!\perp\!\!\!\perp Y$

Recall def We have $X \perp\!\!\!\perp Y$ if

$$\mathbb{P}(X \in A,\ Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

for all sets A, B. Here we will find A, B s.t.

$$\mathbb{P}(X \in A,\ Y \in B) \neq \mathbb{P}(X \in A)\,\mathbb{P}(X \in B)$$

This is enough to prove $\boxed{X \not\perp\!\!\!\perp Y}$

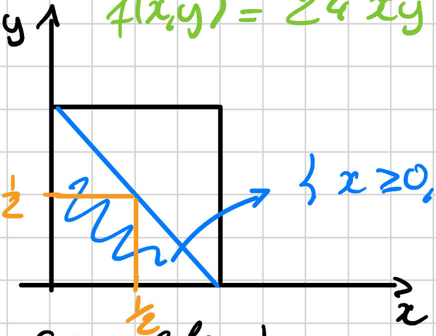
Example of A, B:

$A = [0, \frac{1}{2}]$, $B = [0, \frac{1}{2}]$

Aim : prove

$$\boxed{\mathbb{P}\left(0 \le X \le \tfrac{1}{2},\ 0 \le Y \le \tfrac{1}{2}\right) \ne \mathbb{P}\left(0 \le X \le \tfrac{1}{2}\right)\mathbb{P}\left(0 \le Y \le \tfrac{1}{2}\right)}$$

$$f(x,y) = 24\,xy\;\mathbb{1}_{(0,\infty)^2}(x,y)\,\mathbb{1}_{(x+y\le 1)}$$



$$\{x \ge 0,\ y \ge 0,\ x+y \le 1\}$$

Simplification

$$\mathbb{P}((X,Y) \in [0,\tfrac{1}{2}])$$

$$= \int_{[0,\frac{1}{2}]^2} f(x,y)\,dx\,dy$$

$$= \int_{[0,\frac{1}{2}]^2} 24\,x\,y\,dx\,dy = 24\int_0^{\frac{1}{2}} x\,dx \int_0^{\frac{1}{2}} y\,dy$$

Summary

$$\boxed{\mathbb{P}\left((X,Y) \in [0,\tfrac{1}{2}]^2\right)}$$

$$= 24 \int_0^{\frac{1}{2}} x \, dx \int_0^{\frac{1}{2}} y \, dy$$

$$= 24 \left(\frac{x^2}{2} \Big|_0^{\frac{1}{2}} \right)^2$$

$$\boxed{= \frac{3}{8}}$$

$f(x,y) = 24xy\,\mathbb{1}_{(0,\infty)^2}(x,y)\,\mathbb{1}_{(x+y\leq 1)}$

Compute

$$\boxed{\mathbb{P}(0 \leq X \leq \tfrac{1}{2})}$$

$$= 24 \int_0^{\frac{1}{2}} dx \, x \int_0^{1-x} y \, dy$$

$$= 24 \int_0^{\frac{1}{2}} dx \; x \; \frac{y^2}{2}\Big|_0^{1-x}$$

$$= 12 \int_0^{\frac{1}{2}} x(1-x)^2 \, dx$$

polynomial in x

$$= \cdots \boxed{= \frac{11}{16}}$$

Conclusion

$$\frac{3}{8} = \mathbb{P}\left((X,Y) \in [0,\tfrac{1}{2}]^2\right)$$

$$\neq \mathbb{P}\left(X \in [0,\tfrac{1}{2}]\right) \mathbb{P}\left(Y \in [0,\tfrac{1}{2}]\right)$$

$$= \left(\frac{11}{16}\right)^2$$

$$\Rightarrow \quad \boxed{X \not\perp\!\!\!\perp Y}$$

Example of non independence (2)

Recall density:

$$f(x,y) = 24xy\,\mathbf{1}_{(0,\infty)^2}(x,y)\mathbf{1}_{(0<x+y<1)},$$

Non product structure:
X, Y satisfy the relation: $X + Y < 1$.

Checking non independence: We have

$$\mathbf{P}\left((X,Y) \in \left[0,\frac{1}{2}\right]^2\right) = \int_{[0,\frac{1}{2}]^2} 24xy\,dxdy = \frac{3}{8}$$

and

$$\mathbf{P}\left(X \in \left[0,\frac{1}{2}\right]\right)\mathbf{P}\left(Y \in \left[0,\frac{1}{2}\right]\right) = \left(24\int_0^{\frac{1}{2}} dx\, x \int_0^{1-x} y\,dy\right)^2 = \left(\frac{11}{16}\right)^2$$