Outline

Density of a sum

Proposition 13.

Let

- X, Y continuous random variables
- Hypothesis: $X \perp\!\!\!\perp Y$
- Set $Z = X + Y$

convolution product

Then the density of Z is given by

$$f_Z(a) = [f_X * f_Y](a) = \int_{\mathbb{R}} f_X(a - y) \, f_Y(y) \, dy$$

Proof

Characterization by expectations: Let $\varphi \in \mathcal{C}(\mathbb{R})$. Then

$$\mathbf{E}[\varphi(Z)] = \int_{\mathbb{R}^2} \varphi(x+y) f_X(x) f_Y(y)\, dxdy$$

Change of variable:
$x + y = a$ and $y = b$, thus $J = 1$

Expression for $\mathbf{E}[\varphi(Z)]$:

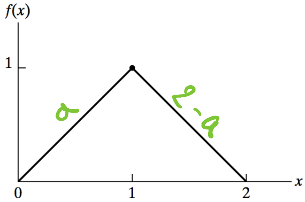
$$\mathbf{E}[\varphi(Z)] = \int_{\mathbb{R}} \varphi(a) \left(\int_{\mathbb{R}} f_X(a-b)\, f_Y(b)\, db \right) da$$

Triangular distribution

Proposition 14.

Let

- $X, Y \sim \mathcal{U}([0,1])$
- Hypothesis: $X \perp\!\!\!\perp Y$
- Set $Z = X + Y$



Then the density of Z is given by

$$f_Z(a) = a\,\mathbf{1}_{[0,1]}(a) + (2-a)\,\mathbf{1}_{[1,2]}(a)$$

Aim: $X, Y \sim \mathcal{U}([0,1])$, $X \perp\!\!\!\perp Y$, $Z = X+Y$. We wish to compute f_Z. According to Prop 13,

$$f_Z(a) = \int_{\mathbb{R}} f_X(a-y)\, \underbrace{f_Y(y)}_{\mathbb{1}_{[0,1]}(y)}\, dy$$

If $Y \sim \mathcal{U}([0,1])$, we have

$$f_Y(y) = \mathbb{1}_{[0,1]}(y)$$

$$\Rightarrow \boxed{f_Z(a) = \int_0^1 f_X(a-y)\, dy}$$

$$f_X(x) = \mathbb{1}_{[0,1]}(x)$$

Summary

$$f_Z(a) = \int_0^1 f_X(a-y)\,dy$$

$= 1$ if $0 \leq a-y \leq 1$

$\iff a-1 \leq y \leq a$

$$\Rightarrow f_Z(a) = \int_0^1 \mathbb{1}_{[a-1,a]}(y)\,dy$$

$$= \text{length}\left([0,1] \cap [a-1,a]\right)$$

check

$$= \begin{cases} a & \text{if } 0 \leq a < 1 \\ 2-a & \text{if } 1 \leq a < 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof

Application of Proposition 13:

$$f_Z(a) = \int_0^1 f_X(a-y)\,dy = \int_{[0,1]\cap[a-1,a]} dy = |[0,1] \cap [a-1,a]|$$

Case 1: $a \in [0,1]$: Then $[0,1] \cap [a-1,a] = [0,a]$ and

$$f_Z(a) = a$$

Case 2: $a \in (1,2]$: Then $[0,1] \cap [a-1,a] = [a-1,1]$ and

$$f_Z(a) = 2 - a$$

Sums of Gamma random variables

Proposition 15.

Let

- $X_1, \ldots, X_n$ independent random variables
- $X_i \sim \Gamma(t_i, \lambda)$
- $Z = \sum_{i=1}^n X_i$

Then

$$Z \sim \Gamma\left(\sum_{i=1}^n t_i, \lambda\right)$$

Remark: This result includes

- Sums of exponential random variables
- Sums of chi-square random variables

Sums of Gaussian random variables

Proposition 16.

Let

- $X_1, \ldots, X_n$ independent random variables
- $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
- $Z = \sum_{i=1}^n X_i$

Then

$$Z \sim \mathcal{N}\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$$

Example: basketball (1)

Situation:

- A basketball team will play a 44-game season
- 26 games are against class A teams, with probability of win $= .4$
- 18 games are against class B teams, with probability of win $= .7$
- Results of the different games are independent.

Question: Approximate the probability that

1. The team wins 25 games or more
2. The team wins more games against class A teams than it does against class B teams

Situation :

(i) We are playing 26 games against type A teams, $P(\text{win}) = .4 = p_A$ (with n_A = 26)

Set X_A = total # wins against type A teams

$$\Rightarrow X_A \sim \text{Bin}(26, .4)$$

Approximation

~~Poisson~~ ? De Moivre ?

$$X_A \approx \mathcal{N}(26 \times .4 ; 26 \times .4 \times .6)$$

$$\boxed{X_A \approx \mathcal{N}(10.4 ; 6.24)}$$

(iii) We are playing 18 games against type B teams, $P(\text{win}) = .7$

Set X_B = total # wins against type B teams

$\Rightarrow X_B \sim \text{Bin}(18, .7)$

Approximation ~~Poisson~~ ? De Moivre ?

$X_B \approx \mathcal{N}(18 \times .7 \; ; \; 18 \times .7 \times .3)$

$X_B \approx \mathcal{N}(12.60 \; ; \; 3.78)$

Example: basketball (2)

Model: We set

- $X_A = \#$ games the team wins against class A
- $X_B = \#$ games the team wins against class B

Then $X_A \perp\!\!\!\perp X_B$ and

$$X_A \sim \text{Bin}(26, 0.4), \quad X_B \sim \text{Bin}(18, 0.7)$$

Approximation for X_A, X_B: According to DeMoivre-Laplace,

$$X_A \approx \mathcal{N}(10.4; 6.24), \qquad X_B \approx \mathcal{N}(12.60; 3.78)$$

Example: basketball (3)

$\mu_A + \mu_B = 10.4 + 12.6$

Approximation for $X_A + X_B$: Since $X_A \perp\!\!\!\perp X_B$, $\sigma_A^2 + \sigma_B^2 = 6.24 + 3.78$

total # wins = $X_A + X_B \approx \mathcal{N}(23; 10.02)$

Question 1: We have

$$
\begin{aligned}
\mathbf{P}\left(X_A + X_B \geq 25\right) &= \mathbf{P}\left(X_A + X_B \geq 24.5\right) \\
&= \mathbf{P}\left(\frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}}\right) \\
&\simeq 1 - \mathbf{P}\left(Z < .47\right) = 1 - \phi(.47) \\
&\simeq 1 - .6808 \\
&\simeq .32
\end{aligned}
$$

(integer; $\frac{24.5-23}{\sqrt{10.02}} = .47$)

$Z \sim \mathcal{N}(0,1)$

$= \mathbb{P}(Z \geq .47)$

Example: basketball (4)

Approximation for $X_A - X_B$: Since $X_A \perp\!\!\!\perp X_B$,

$$X_A - X_B \approx \mathcal{N}(-2.2; 10.02)$$

Question 2: We have

$$\begin{aligned}
\mathbf{P}(X_A - X_B > 0) &= \mathbf{P}(X_A - X_B \geq .5) \\
&= \mathbf{P}\left(\frac{X_A - X_B + 2.2}{\sqrt{10.02}} \geq \frac{.5 + 2.2}{\sqrt{10.02}}\right) \\
&\simeq 1 - \mathbf{P}(Z < .8530) \\
&\simeq .1968
\end{aligned}$$