Sums of Poisson random variables

Proposition 17.

Let

- $X_1, \ldots, X_n$ independent random variables
- $X_i \sim \mathcal{P}(\lambda_i)$
- $Z = \sum_{i=1}^n X_i$

Then

$$Z \sim \mathcal{P}\left(\sum_{i=1}^n \lambda_i\right)$$

Consider $X_1 \sim \mathcal{P}(\lambda_1)$ $X_1 \perp X_2$, $X_2 \sim \mathcal{P}(\lambda_2)$, $Z = X_1 + X_2$

$$\boxed{\mathbb{P}(Z=n)} = \mathbb{P}(X_1+X_2=n) = \mathbb{P}\Big(\overbrace{\bigcup_{k=0}^{n}(X_1=k,\ X_2=n-k)}^{\text{disjoint}}\Big)$$

$$= \sum_{k=0}^{n} \mathbb{P}(X_1=k,\ X_2=n-k) \overset{\perp}{=} \sum_{k=0}^{n} \mathbb{P}(X_1=k)\,\mathbb{P}(X_2=n-k)$$

$$= \frac{1}{n!}\sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}\, e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}\; n!$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\binom{n}{k}\lambda_1^k\,\lambda_2^{n-k}$$

$$\overset{\text{Binomial}}{=} \boxed{\frac{e^{-(\lambda_1+\lambda_2)}}{n!}(\lambda_1+\lambda_2)^n}$$

$$\Rightarrow \boxed{Z \sim \mathcal{P}(\lambda_1+\lambda_2)}$$

Proof for 2 random variables

Hypothesis:
$X_1 \sim \mathcal{P}(\lambda_1)$, $X_2 \sim \mathcal{P}(\lambda_2)$ and $X_1 \perp\!\!\!\perp X_2$

Computation: For $n \geq 0$,

$$\begin{aligned}
\mathbf{P}(X_1 + X_2 = n) &= \sum_{k=0}^{n} \mathbf{P}(X_1 = k)\, \mathbf{P}(X_2 = n-k) \\
&= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!}\, e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}
\end{aligned}$$

Sums of Binomial random variables

Proposition 18.

Let

- $X_1, \ldots, X_n$ independent random variables
- $X_i \sim \text{Bin}(n_i, p)$
- $Z = \sum_{i=1}^n X_i$

Then

$$Z \sim \text{Bin}\left(\sum_{i=1}^n n_i, p\right)$$

Outline

General definition

Definition 19.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y
- y such that $p_Y(y) > 0$

Then the conditional pmf of X given $Y = y$ is defined by

$$p_{X|Y}(x|\,y) = \mathbf{P}\,(\overbrace{X = x}^{A}|\,\overbrace{Y = y}^{B}) = \frac{p(x, y)}{p_Y(y)}$$

Example ctd: tossing 3 coins (1)

Experiment:
Tossing a coin 3 times

Events: We consider

$$A = \text{"At most one Head"}$$
$$B = \text{"At least one Head and one Tail"}$$

Random variables: Set

$$X_1 = \mathbf{1}_A, \qquad X_2 = \mathbf{1}_B, \qquad X = (X_1, X_2)$$

Example ctd: tossing 3 coins (2)

We have seen:

$X_1 \backslash X_2$	0	1	Marg. X_1
0	1/8	3/8	1/2
1	1/8	3/8	1/2
Marg. X_2	1/4	3/4	1

Conditional probabilities given $X_1 = 0$:

$$\frac{p(0,0)}{p_{X_1}(0)} = p_{X_2|X_1}(0|\,0) = \frac{1/8}{1/2} = \frac{1}{4}, \quad \frac{p(0,1)}{p_{X_1}(0)} = p_{X_2|X_1}(1|\,0) = \frac{3/8}{1/2} = \frac{3}{4}$$

$Law(X_2 | X_1 = 0) = \mathcal{B}(3/4) = Law(X_2) \Rightarrow$ due to $X_1 \perp\!\!\!\perp X_2$

Conditional probabilities given $X_2 = 1$:

$$\frac{p(0,1)}{p_{X_2}(1)} = p_{X_1|X_2}(0|\,1) = \frac{3/8}{3/4} = \frac{1}{2}, \quad p_{X_1|X_2}(1|\,1) = \frac{3/8}{3/4} = \frac{1}{2}$$

$\mathcal{L}(X_1 | X_2 = 1) = \mathcal{B}(1/2) = \mathcal{L}(X_1)$

Conditioning Poisson random variables

Proposition 20.

Let

- $X \sim \mathcal{P}(\lambda_1)$, $Y \sim \mathcal{P}(\lambda_2)$
- $X \perp\!\!\!\perp Y$
- $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

Then

$$\mathcal{L}\left(X \middle| X+Y=n\right) = \text{Bin}(n, p)$$

Aim: find $\mathcal{L}(X \mid X+Y=n)$, that is

$$\boxed{\mathbb{P}(X=k \mid X+Y=n) \qquad , k=0,\dots,n}$$

$$= \frac{\mathbb{P}(X=k,\ X+Y=n)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{\mathbb{P}(X=k,\ Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$$\overset{\perp\!\!\!\perp}{=} \frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$X \sim \mathcal{P}(\lambda_1) \quad Y \sim \mathcal{P}(\lambda_2)$

$X+Y \sim \mathcal{P}(\lambda_1+\lambda_2)$

We compute

$p = \dfrac{\lambda_1}{\lambda_1+\lambda_2}$

$$\frac{\mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)}{\mathbb{P}(X+Y=n)}$$

$$= \frac{e^{-\lambda_1}\lambda_1^k \; e^{-\lambda_2}\lambda_2^{n-k} \; n!}{k!\,(n-k)!\; e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}$$

$$= \binom{n}{k} \frac{\lambda_1^k}{(\lambda_1+\lambda_2)^k} \frac{\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n-k}}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$\Rightarrow \quad \boxed{\mathcal{L}(X \mid X+Y=n) = \mathrm{Bin}(n,p)}$$

Proof (1)

Expression for the conditional probabilities:
Let $0 \leq k \leq n$. Then invoking $X \perp\!\!\!\perp Y$,

$$\mathbf{P}\left(X = k \middle| X + Y = n\right) = \frac{\mathbf{P}\left(X = k\right)\mathbf{P}\left(Y = n - k\right)}{\mathbf{P}\left(X + Y = n\right)}$$

Law of $X + Y$: We have seen

$$X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$$

Proof (2)

Computation of the conditional probabilities:

$$
\begin{aligned}
&\mathbf{P}\left(X = k \,|\, X + Y = n\right) \\
&= e^{-\lambda_1}\frac{\lambda_1^k}{k!} e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}\left[e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} \\
&= \binom{n}{k} p^k (1-p)^{n-k}
\end{aligned}
$$

Conclusion:

$$
\mathcal{L}\left(X \,|\, X + Y = n\right) = \mathrm{Bin}(n, p)
$$

Outline

General definition

Definition 21.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y
- y such that $f_Y(y) > 0$

Then the conditional density of X given $Y = y$ is defined by

$$f_{X|Y}(x|\,y) = \frac{f(x, y)}{f_Y(y)}$$

Justification of the definition

Heuristics: $f_{X|Y}(x|\,y)$ can be interpreted as

$$\begin{aligned} f_{X|Y}(x|\,y)\,dx &= \frac{f(x,y)\,dxdy}{f_Y(y)\,dy} \\ &\simeq \frac{\mathbf{P}\left(x \le X \le x+dx,\; y \le Y \le y+dy\right)}{\mathbf{P}\left(y \le Y \le y+dy\right)} \\ &= \mathbf{P}\left(x \le X \le x+dx|\, y \le Y \le y+dy\right) \end{aligned}$$

Use of the conditional probability: compute probabilities like

$$\mathbf{P}\left(X \in A|\, Y = y\right) = \int_A f_{X|Y}(x|\,y)\,dx$$

Rigorous definition: see MA 539