Outline

General definition

Definition 21.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y
- y such that $f_Y(y) > 0$

Then the conditional density of X given $Y = y$ is defined by

$$f_{X|Y}(x|\,y) = \frac{f(x,y)}{f_Y(y)}$$

Justification of the definition

Heuristics: $f_{X|Y}(x|\,y)$ can be interpreted as

$$
\begin{aligned}
f_{X|Y}(x|\,y)\,dx &= \frac{f(x,y)\,dxdy}{f_Y(y)\,dy} \\
&\simeq \frac{\mathbf{P}\left(x \le X \le x+dx,\; y \le Y \le y+dy\right)}{\mathbf{P}\left(y \le Y \le y+dy\right)} \\
&= \mathbf{P}\left(x \le X \le x+dx|\, y \le Y \le y+dy\right)
\end{aligned}
$$

Use of the conditional probability: compute probabilities like

$$\mathbf{P}\left(X \in A|\, Y = y\right) = \int_A f_{X|Y}(x|\,y)\,dx$$

Rigorous definition: see MA 539

Simple example of continuous conditioning (1)

Density: Let (X, Y) be a random vector with density

$$f(x,y) = \frac{e^{-\frac{x}{y}} e^{-y}}{y} \mathbf{1}_{(0,\infty)}(x) \mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

$$\mathbf{P}(X > 1 | Y = y)$$

Strategy: (i) Compute $f_Y(y)$

(ii) compute $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

(iii) Compute $\mathbb{P}(X>1 | Y=y) = \int_1^\infty f_{X|Y}(x|y)\, dx$

$$f(x,y) = \frac{e^{-x/y} e^{-y}}{y} \mathbb{1}_{(0,\infty)}(x) \mathbb{1}_{(0,\infty)}(y)$$

Step (i) Compute

$\int_{\mathbb{R}} f(x)\,dx = 1$ for f density

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$$= \int_{\mathbb{R}} \frac{e^{-x/y}}{y} e^{-y} \mathbb{1}_{(0,\infty)}(x) \mathbb{1}_{(0,\infty)}(y)\,dx$$

$$= e^{-y} \mathbb{1}_{(0,\infty)}(y) \int_0^\infty \frac{1}{y} e^{-x/y}\,dx$$

Density of r.v $\mathcal{E}(\frac{1}{y})$

$$\Rightarrow f_Y(y) = e^{-y} \mathbb{1}_{(0,\infty)}(y)$$

Rmk $Y \sim \mathcal{E}(1)$

Recalling the main densities

Law	Density
$\mathcal{U}([a,b])$	$\frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$
$\mathcal{E}(\lambda)$	$\lambda e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x)$
$\mathcal{N}(0,1)$	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$
$\mathcal{N}(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$f(x,y) = \frac{e^{-x/y}\, e^{-y}}{y} \mathbb{1}_{(0,\infty)}(x)\, \mathbb{1}_{(0,\infty)}(y)$$

Step (ii) Compute, if $x, y > 0$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} \quad e^{-y}$$

$$= \frac{e^{-x/y}\, e^{-y}}{y\, e^{-y}}$$

$$= \begin{cases} \dfrac{e^{-x/y}}{y} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Rmk $\mathcal{L}(X \mid Y = y) = \mathcal{E}\left(\frac{1}{y}\right)$

Step (iii) We wish to compute

$$\mathbb{P}(X > 1 \mid Y = y)$$

$$= \int_1^\infty f_{X|Y}(x|y)\,dx$$

$$= \int_1^\infty \frac{1}{y} e^{-x/y}\,dx$$

$$= \mathbb{P}(Z > 1), \quad \text{with } Z \sim \mathcal{E}\left(\frac{1}{y}\right)$$

$$= e^{-\frac{1}{y}}$$ (Formula for tails of $\mathcal{E}(\frac{1}{y})$)

Otherwise: $\mathbb{P}(X > 1 \mid Y = y)$

$$= -e^{-x/y}\Big|_1^\infty = e^{-\frac{1}{y}}$$

Simple example of continuous conditioning (2)

Marginal distribution of Y: We have

$$\begin{aligned} f_Y(y) &= \int_0^\infty f(x,y)\,dx \\ &= \frac{e^{-y}}{y}\left(\int_0^\infty e^{-\frac{x}{y}}\,dx\right)\mathbf{1}_{(0,\infty)}(y) \\ &= e^{-y}\,\mathbf{1}_{(0,\infty)}(y) \end{aligned}$$

Conditional density: For $y > 0$ we have

$$f_{X|Y}(x|\,y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}}}{y}\,\mathbf{1}_{(0,\infty)}(x)$$

Namely $\mathcal{L}(X|\,Y = y) = \mathcal{E}(\frac{1}{y})$

Simple example of continuous conditioning (3)

Conditional probability:

$$\begin{aligned}\mathbf{P}(X > 1 | Y = y) &= \int_1^\infty f_{X|Y}(x|y)\,dx \\ &= \int_1^\infty \frac{e^{-\frac{x}{y}}}{y}\,dx \\ &= e^{-\frac{1}{y}}\end{aligned}$$

Outline

Characterizing r.v by expected values

Notation:

$C_b(\mathbb{R}^2) \equiv$ set of continuous and bounded functions on $\mathbb{R}^2$.

Theorem 22.

Let $X = (X_1, X_2)$ be a r.v in $\mathbb{R}^2$. We assume that

$$\mathbf{E}[\varphi(X_1, X_2)] = \int_{\mathbb{R}^2} \varphi(x_1, x_2)\, f(x_1, x_2)\, dx_1 dx_2,$$

for all functions $\varphi \in C_b(\mathbb{R}^2)$.

Then (X_1, X_2) is continuous, with density f.

Application: change of variable

Problem: Let

- $X = (X_1, X_2)$ random variable with density f.
- Set $Y = h(X)$ with $h : \mathbb{R}^2 \to \mathbb{R}^2$.

We wish to find the density of Y.

Application: change of variable (2)

Recipe: One proceeds as follows

1. For $\varphi \in C_b(\mathbb{R}^2)$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(h(X))] = \int_{\mathbb{R}^2} \varphi(h(x_1, x_2))\, f(x_1, x_2)\, dx_1 dx_2.$$

2. Change variables $y = h(x)$ in the integral.
   After some elementary computations we get

$$\mathbf{E}[\varphi(Y)] = \int_{\mathbb{R}^2} \varphi(y_1, y_2)\, g(y_1, y_2)\, dy_1 dy_2.$$

3. This characterizes Y, which admits a density g

Polar coordinates of Gaussian vectors (1)

Standard Gaussian vector in $\mathbb{R}^2$: Consider

- $X, Y \sim \mathcal{N}(0,1)$, with $X \perp\!\!\!\perp Y$
- $Z = (X, Y)$

Polar coordinates: Set

$$(X, Y) = (R\cos(\Theta), R\sin(\Theta))$$

Question:
Find the joint density of (R, Θ)

Model

$X \sim N(0,1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$Y \sim N(0,1) \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$X \perp\!\!\!\perp Y$

$\Rightarrow Z = (X, Y)$ has density

$$\boxed{f(x,y)} = f_X(x)\, f_Y(y)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$\boxed{= \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)}$$

$$f(x,y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

General formula

We set $(x = r\cos(\theta), \; y = r\sin(\theta)) = g^{-1}(r,\theta)$

Then the density in (r,θ) is

$$h(r,\theta) = f(g^{-1}(r,\theta))\, J(r,\theta)\, \mathbb{1}((r,\theta) = g(x,y))$$

Here

$\cos^2(\theta) + \sin^2(\theta) = 1$

- $$f(g^{-1}(r,\theta)) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(r^2\cos^2\theta + r^2\sin^2\theta)\right)$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2}$$

- $x = (r\cos(\theta),\ r\sin(\theta))$

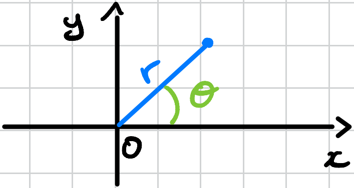
$$\Rightarrow J(r,\theta) = \left| \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right|$$

$$= \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\,(\cos^2(\theta) + \sin^2(\theta))$$

$$= \boxed{r = J(r,\theta)}$$

- **Recall** $x = r\cos(\theta)$ $y = r\sin(\theta)$



We get $r > 0$

$\theta \in (0, 2\pi)$

Conclusion. The density for (R, Θ) is

$$h(r,\theta) = \frac{1}{2\pi} e^{-\frac{r^2}{2}} \times r \times \mathbb{1}_{(0,2\pi)}(\theta)\, \mathbb{1}_{(0,\infty)}(r)$$

Polar coordinates of Gaussian vectors (2)

Decomposition of the expected value: For $\varphi \in \mathcal{C}_b(\mathbb{R}^2)$,

$$\begin{aligned} \mathbf{E}\left[\varphi(R,\Theta)\right] &= \mathbf{E}\left[\varphi(R,\Theta)\,\mathbf{1}_{(Y>0)}\right] + \mathbf{E}\left[\varphi(R,\Theta)\,\mathbf{1}_{(Y<0)}\right] \\ &\equiv A_+ + A_- \end{aligned}$$

Expression for A_+:

$$\begin{aligned} A_+ &= \mathbf{E}\left[\varphi\left((X^2+Y^2)^{1/2}, \tan^{-1}\left(\frac{Y}{X}\right)\right)\mathbf{1}_{(Y>0)}\right] \\ &= \int_{\mathbb{R}\times\mathbb{R}_+} \varphi\left((x^2+y^2)^{1/2}, \tan^{-1}\left(\frac{y}{x}\right)\right) \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\, dxdy \end{aligned}$$

Polar coordinates of Gaussian vectors (3)

Change of variable for A_+: Set

$$x = r\cos(\theta), \qquad y = r\sin(\theta) \quad \Longrightarrow \quad J(r,\theta) = r$$

Then

$$A_+ = \int_{\mathbb{R}_+\times(0,\pi)} \varphi(r,\theta)\,\frac{r\,e^{-\frac{r^2}{2}}}{2\pi}\,drd\theta$$

Change of variable for A_-: We find

$$A_- = \int_{\mathbb{R}_+\times(\pi,2\pi)} \varphi(r,\theta)\,\frac{r\,e^{-\frac{r^2}{2}}}{2\pi}\,drd\theta$$

Polar coordinates of Gaussian vectors (4)

Expression for the expected value:

$$\mathbf{E}\left[\varphi(R,\Theta)\right] = \int_{\mathbb{R}_+\times(0,2\pi)} \varphi(r,\theta)\, \frac{r\, e^{-\frac{r^2}{2}}}{2\pi}\, dr d\theta$$

Joint density for (R,Θ):

$$f(r,\theta) = \frac{1}{2\pi}\, \mathbf{1}_{(0,2\pi)}(\theta) \times r\, e^{-\frac{r^2}{2}}\, \mathbf{1}_{\mathbb{R}_+}(r)$$

Otherwise stated:

- $R \sim$ Rayleigh, $\Theta \sim \mathcal{U}([0,2\pi])$
- $R \perp\!\!\!\perp \Theta$

Change of variable: general result (repeated)

Theorem 23.

Let

- $X = (X_1, X_2)$ continuous random variable
- Density: f_X → joint density
- g diffeomorphism of $\mathbb{R}^2$
- $Y = g(X)$

Then Y has a density f_Y given by

$$f_Y(y) = f_X\left(g^{-1}(y)\right) J(y)\, \mathbf{1}_{\{y=g(x) \text{ for some } x\}}$$