Outline

Global objective

Aim: Introduce

- Sample space
- Events of an experiment
- Probability of an event
- Show how probabilities can be computed in certain situations

Outline

Sample space

Situation: We run an experiment for which

- Specific outcome is unknown
- Set S of possible outcomes is known

Terminology:

In the context above S is called sample space

Experiment 1: Roll 1 dice. Then

S = set of possible outcomes (sample space)

$= \{1, 2, 3, 4, 5, 6\}$

Experiment 2: Roll 2 dice. Then

$S = \{(1,1); (1,2); (1,3); \dots (6,6)\}$

$= \{(i,j);\ i,j \in \{1, \dots, 6\}\}$

$= \{1, 2, \dots, 6\}^2$

Experiment 3: Observe the lifetime of a light bulb

$$S = \{ x \in \mathbb{R} \, ; \, x \geq 0 \}$$

$$= [0, \infty)$$

$$= \mathbb{R}_+$$

Examples of sample spaces

Tossing two dice: We have

$$\begin{aligned} S &= \{1,2,3,4,5,6\}^2 \\ &= \{(i,j);\ i,j = 1,2,3,4,5,6\} \end{aligned}$$

Lifetime of a transistor: We have

$$S = \mathbb{R}_+ = \{x \in \mathbb{R};\ 0 \le x < \infty\}$$

Events

Definition 1.

Consider

- Experiment with sample space S
- A subset E of S

Then

E is called event

Event for rolling 2 dice

$S = \{1, .., 6\}^2$

E = " Sum of 2 dice = 6 "

$= \{ (1,5), (5,1), (2,4), (4,2), (3,3) \}$

$= \{ (i,j) ; \ i,j \in \{1, .., 6\} \text{ s.t. } i+j = 6 \}$

$\subset S$

Thus E is an event

Event for lifetime of a bulb

$S = [0, \infty)$

E = " bulb will live more than 7 hours "

$= \{ x \in [0, \infty) \, ; \, x > 7 \}$

$= (7, \infty)$

$\subset S$

E is an event

Example of event (1)

Tossing two dice: We have

$$S = \{1, 2, 3, 4, 5, 6\}^2$$

Event: We define

$$E = (\text{Sum of dice is equal to 7})$$

Example of event (2)

Description of E as a subset:

$$E = \{(1,6);(2,5);(3,4);(4,3);(5,2);(6,1)\} \subset S$$

Second example of event (1)

Lifetime of a transistor: We have

$$S = \mathbb{R}_+ = \{x \in \mathbb{R};\, 0 \le x < \infty\}$$

Event: We define

$$E = (\text{Transistor does not last longer than 5 hours})$$

Second example of event (2)

Description of E as a subset:

$$E = [0, 5] \subset S$$

Operations on events

Complement: E^c is the set of elements of S not in E

Two dice example:

$$E^c = \text{"Sum of two dice different from 7"}$$

Union, Intersection: For the two dice example, if

$$B = \text{"Sum of two dice is divisible by 3"}$$
$$C = \text{"Sum of two dice is divisible by 4"}$$

Then

$$B \cup C = \text{"Sum of two dice is divisible by 3 or 4"}$$
$$B \cap C = BC = \text{"Sum of two dice is divisible by 3 and 4"}$$

Example of complement for 2 dice

$E =$ " sum of dice $= 6$ "

Then $E^c =$ " sum of dice $\neq 6$ "

$= \{ (i,j) \, ; \; i,j \in \{1,\dots,6\} \, , \; i+j \neq 6 \}$

$= S \setminus \{ (1,5); (2,4); (3,3); (4,2); (5,1) \}$

$\subset S$

Example of ∪ for 2 dice

$S = \{1, \dots, 6\}^2$

B = " sum divisible by 3 "
C = " " " " 4 "

Then $B \cup C$ = " sum div. by 3 or 4 "

$= \{(i,j) ;\ i,j \in \{1,..,6\} \text{ s.t } i+j \in \{3,4,6,8,9,12\}\}$

Example of $\cap$ for 2 dice

$S = \{1, \dots, 6\}^2$

B = " sum divisible by 3 "

C = " " " " 4 "

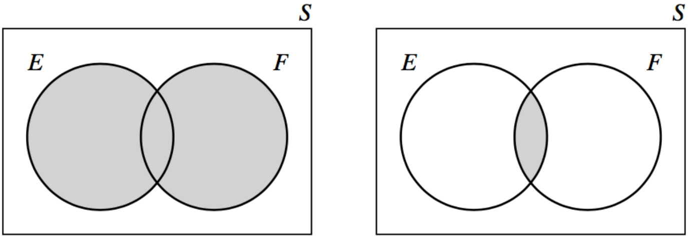
Then $\boxed{B \cap C}$ = " sum div. by 3 and 4 "

$= \{(i,j) ;\ i,j \in \{1,\dots,6\} \text{ s.t } i+j = 12\}$

$\boxed{= \{(6,6)\}}$

Illustration (1)

Union and intersection:

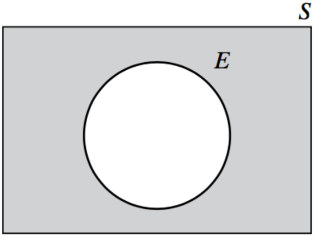


(a) Shaded region: $E \cup F$.

(b) Shaded region: EF.

Illustration (2)

Complement:



(c) Shaded region: E^c.

Illustration (3)

Subset:

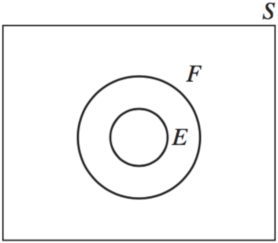


Figure: $E \subset F$

Laws for elementary operations

Commutative law:

$$E \cup F = F \cup E, \qquad EF = FE$$

Associative law:

$$(E \cup F) \cup G = E \cup (F \cup G), \qquad E(FG) = (EF)G$$

Distributive laws:

$$\begin{aligned} (E \cup F)G &= EG \cup FG \\ (EF) \cup G &= (E \cup G)(F \cup G) \end{aligned}$$

Illustration

Distributive law:

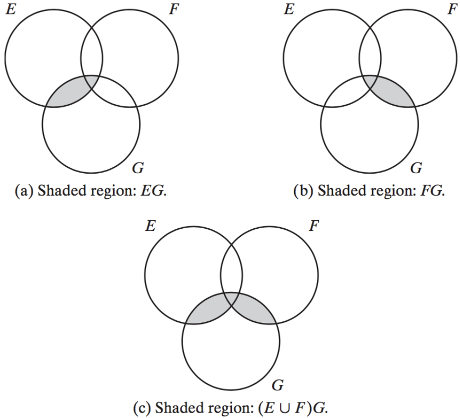


Figure: $(E \cup F)G = EG \cup FG$

De Morgan's laws

Proposition 2.

Let

- S sample space
- $E_1, \ldots, E_n$ events

For $n=2$

$(F \cup G)^c = F^c \cap G^c$

$(F \cap G)^c = F^c \cup G^c$

Then

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

Proof (1)

Proof of $(\cup_{i=1}^{n} E_i)^c \subset \cap_{i=1}^{n} E_i^c$:

Assume $x \in (\cup_{i=1}^{n} E_i)^c$ Then

$$\begin{aligned} x \notin \cup_{i=1}^{n} E_i &\Longrightarrow \text{for all } i \leq n,\ x \notin E_i \\ &\Longrightarrow \text{for all } i \leq n,\ x \in E_i^c \\ &\Longrightarrow x \in \cap_{i=1}^{n} E_i^c \end{aligned}$$

Proof (2)

Proof of $\cap_{i=1}^n E_i^c \subset (\cup_{i=1}^n E_i)^c$:

Assume $x \in \cap_{i=1}^n E_i^c$ Then

$$\begin{aligned} \text{for all } i \le n,\ x \in E_i^c &\implies \text{for all } i \le n,\ x \notin E_i \\ &\implies x \notin \cup_{i=1}^n E_i \\ &\implies x \in (\cup_{i=1}^n E_i)^c \end{aligned}$$