Outline

Cond. pmf in the discrete case (repeated)

Definition 24.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y
- y such that $p_Y(y) > 0$

Then the conditional pmf of X given $Y = y$ is defined by

$$p_{X|Y}(x|\,y) = \mathbf{P}\left(X = x|\,Y = y\right) = \frac{p(x,y)}{p_Y(y)}$$

Cond. expectation in the discrete case

Definition 25.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y, y such that $p_Y(y) > 0$
- $p_{X|Y}(x|\,y)$ conditional distribution

Then the conditional exp. of X given $Y = y$ is defined by

$$\mathbf{E}\left[X|\,Y = y\right] = \sum_{x \in \mathcal{E}} x\, p_{X|Y}(x|\,y)$$

compare with

$$\mathbb{E}[X] = \sum_{x \in \mathcal{E}} x\, p_X(x)$$

Binomial example (1)

Situation: Let

- $X, Y \sim \text{Bin}(n, p)$, $X \perp\!\!\!\perp Y$
- $Z = X + Y$ $\Rightarrow$ $Z \sim \text{Bin}(2n, p)$

Problem: We wish to compute

$$\mathbf{E}[X | Z = m] \leq m$$

mp ?

- 1st step: compute $p_{X|Z}(k|m)$
- Then $E[X|Z=m] = \sum_{k=0}^{m} k \, p_{X|Z}(k|m)$

Binomial example (2)

$X \sim \text{Bin}(n,p)$
$Y \sim \text{Bin}(n,p)$
$Z \sim \text{Bin}(2n,p)$

Distribution for Z:

$$Z = \sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \sim \text{Bin}(2n, p)$$

Computation for conditional pmf: For $k \leq \min(n, m)$ we have

$$\underbrace{p_{X|Z}(k|m)}_{=}$$

$$\mathbf{P}(X = k | Z = m) = \frac{\mathbf{P}(X = k, \overbrace{X + Y}^{Z} = m)}{\mathbf{P}(Z = m)}$$

$$= \frac{\mathbf{P}(X = k, Y = m - k)}{\mathbf{P}(Z = m)} \overset{\perp}{=} \frac{\mathbf{P}(X=k)\,\mathbf{P}(Y=m-k)}{\mathbf{P}(Z=m)}$$

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \binom{n}{m-k} p^{m-k} (1-p)^{n-(m-k)}}{\binom{2n}{m} p^m (1-p)^{2n-m}}$$

$\mathbb{E}[X \mid X+Y=m]$

Binomial example (3)

2nd step: compute $\sum_{k=0}^{m} k \, p_{X|Z}(k|m)$

Conditional pmf: For $k \leq \min(n, m)$ we have

$$p_{X|Z}(k|\, m) = \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

Recall: If $V \sim \text{HypG}(n, N, m)$ then

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

Identification of the conditional pmf: We have

$$p_{X|Z}(k|\, m) = \text{Pmf of HypG}(\cancel{2n, m, n}) \quad (m, 2n, n)$$

Binomial example (4)

Conditional expectation: Let $V \sim \text{HypG}(\cancel{2n, m}, n)$. Then $m, 2n, n$

$$\mathbf{E}[X|\, Z = m] = \mathbf{E}[V] = m \times \frac{n}{2n}$$

Numerical value:
According to the values for hypergeometric distributions,

$$\mathbf{E}[X|\, Z = m] = m \times \frac{n}{2n} = \frac{m}{2}$$

$$E[X \mid X+Y=m] = \frac{m}{2}$$

Cond. density in the continuous case (repeated)

Definition 26.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y
- y such that $f_Y(y) > 0$

Then the conditional density of X given $Y = y$ is defined by

$$f_{X|Y}(x|\,y) = \frac{f(x,y)}{f_Y(y)}$$

Cond. expectation in the continuous case

Definition 27.

Let

- (X, Y) couple of continuous random variables
- Joint density f
- Marginal densities f_X, f_Y, y such that $f_Y(y) > 0$
- $f_{X|Y}(x|\,y)$ conditional density

Then the conditional exp. of X given $Y = y$ is defined by

$$\mathbf{E}\left[X|\,Y = y\right] = \int_{\mathbb{R}} x\, f_{X|Y}(x|\,y)\, dx$$

Compare to

$$E[X] = \int_{\mathbb{R}} x\, f_X(x)\, dx$$

Example of continuous conditional expectation (1)

Density: Let (X, Y) be a random vector with density

$$\frac{e^{-\frac{x}{y}} e^{-y}}{y} \mathbf{1}_{(0,\infty)}(x) \mathbf{1}_{(0,\infty)}(y)$$

Question: Compute

$$\mathbf{E}[X | Y = y]$$

1st step: Compute $f_{X|Y}(x|y)$

2nd step: $\mathbf{E}[X|Y=y] = \int_{\mathbb{R}} x \, f_{X|Y}(x|y) \, dx$

Example of continuous conditional expectation (2)

Conditional density: For $y > 0$ we have seen that density of $\mathcal{E}(\frac{1}{y})$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{e^{-\frac{x}{y}}}{y} \mathbf{1}_{(0,\infty)}(x)$$

Namely $\mathcal{L}(X|Y=y) = \mathcal{E}(\frac{1}{y})$

If $z \sim \mathcal{E}(\lambda)$, then $\mathbb{E}[z] = \frac{1}{\lambda}$

Conditional expectation: We have

$$\begin{aligned}
\mathbf{E}[X|Y=y] &= \int_{\mathbb{R}} x\, f_{X|Y}(x|y)\, dx \\
&= \int_0^\infty x\, \frac{e^{-\frac{x}{y}}}{y}\, dx \\
&= y
\end{aligned}$$

Or $\mathbb{E}[X|Y=y] = \mathbb{E}[V]$ with $V \sim \mathcal{E}(\frac{1}{y})$

$= \frac{1}{1/y} = y$

Expectation and conditioning

Proposition 28.

Let X, Y be two random variables. Then

1. If X, Y are discrete we have
$$\mathbf{E}[X] = \sum_{y} \mathbf{E}\left[X \middle| Y = y\right] p_Y(y)$$
2. If X, Y are continuous we have
$$\mathbf{E}[X] = \int_{\mathbb{R}} \mathbf{E}\left[X \middle| Y = y\right] f_Y(y)\, dy$$
3. ~~Unified notation~~:
$$\cancel{\mathbf{E}[X] = \mathbf{E}\left\{\mathbf{E}\left[X \middle| Y\right]\right\}}$$

Example: sales in a store (1)

Situation:

We consider a store on a given day. We assume

- $\#$ of people entering in the store has mean 50
- Amount of money spent by each person is \$8 on average
- Indep. between $\#$ persons entering and amount of money spent

Question:

Expected amount of money spent in the store on a given day?

Example: sales in a store (2)

Notation: We set

- $N = \#$ of customers entering the store
- $X_i =$ Amount spent by i-th customer, for $i \geq 1$
- $Z =$ Total amount spent

Expression for Z: We have (double randomness)

randomness #1

$$Z = \sum_{i=1}^{N} X_i$$

randomness #2

Hypothesis: X_i's are i.i.d (independent and identically distributed)

- X_i's follow the same distribution X
- $(X_i)_{i \geq 1} \perp\!\!\!\perp N$

Example: sales in a store (3)

$Z = \sum_{i=1}^{N} X_i$

Computation:

$$
\begin{aligned}
\mathbf{E}[Z] &= \mathbf{E}\left\{\mathbf{E}\left[\cancel{\sum_{i=1}^{N} X_i} \mid N\right]\right\} \\
&= \sum_{n=1}^{\infty} \mathbf{E}\left[\sum_{i=1}^{N} X_i \mid N=n\right] p_N(n) \quad N=n \text{ if we condition} \\
&= \sum_{n=1}^{\infty} \mathbf{E}\left[\sum_{i=1}^{n} X_i \mid N=n\right] p_N(n) \\
&= \sum_{n=1}^{\infty} \sum_{i=1}^{n} \underbrace{\mathbf{E}[X_i \mid N=n]}_{= \mathbb{E}[X_i] \text{ if } X_i \perp\!\!\!\perp N} p_N(n) \\
&= \sum_{n=1}^{\infty} n\, \overset{\text{does not depend on } i}{\mathbf{E}[X]}\, p_N(n) \quad \leftarrow \mathbb{E}[X] \sum_{n=1}^{\infty} n\, p_N(n) \\
&= \mathbf{E}[N]\,\mathbf{E}[X] = 50 \times 8 = 400
\end{aligned}
$$