Limit theorems

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Probability - MA 416

Mostly taken from *A first course in probability* by S. Ross

Outline

1. Weak law of large numbers
2. Central limit theorem

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Weak law of large numbers

Theorem 1.

Consider

independent and identically distributed

- A sequence $\{X_i;\ i \geq 1\}$ of i.i.d random variables
- Write $\mathbf{E}[X_i] = \mu$
- Set

empirical mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mathbf{P}\left(|\bar{X}_n - \mu| > \varepsilon\right) = 0$$

As $n \to \infty$, empirical mean is close to the theoretical mean

Application We flip a fair coin $n = 400$ times. We set

$$X_i = \mathbb{1}(i\text{-th flip is } H)$$

Hyp: X_i's are iid with

$$X_i \sim \mathcal{B}(p = \tfrac{1}{2}), \quad \mathbb{E}[X_i] = p = \frac{1}{2}$$

Then LLN says

$$\mathbb{P}\left(\left| \frac{1}{400} \sum_{i=1}^{400} X_i - \frac{1}{2} \right| > 0.05 \right)$$

is "small" → Pb: Not a very precise statement

With CLT: We will get statements like

$$\mathbb{P}\left(\sum_{i=1}^{400} X_i > 210 \right) \simeq \text{a probability}$$

that can be computed

Outline

1. Weak law of large numbers

2. Central limit theorem

↘ = important

DeMoivre-Laplace theorem (repeated)

Theorem 2.

Let

- $n \geq 1$, $p \in (0, 1)$
- $X_n \sim \text{Bin}(n, p)$
- $a < b$

Rmk : $X_n = \sum_{i=1}^{n} Y_i$, with Y_i i.i.d with $Y_i \sim \mathcal{B}(p)$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(a < \frac{X_n - \overbrace{np}^{=\mathbb{E}[X_n]}}{(\underbrace{np(1-p)}_{\mathrm{Var}(X_n)})^{1/2}} < b \right) = \Phi(b) - \Phi(a)$$

Another way to write De Moivre's theorem

Theorem 3.

Consider

- A sequence $\{Y_i;\ i \geq 1\}$ of indep. $\mathcal{B}(p)$ random variables
- We have $\mathbf{E}[Y_i] = p$ and $\mathbf{Var}(Y_i) = p(1-p) \equiv \sigma^2$
- Set

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{X_n}{n}$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbf{P}\left(a < \sqrt{n}\left(\frac{\bar{Y}_n - p}{\sigma}\right) < b\right) = \Phi(b) - \Phi(a)$$

Central limit theorem

Theorem 4.

Consider

- A sequence $\{X_i;\ i \geq 1\}$ of i.i.d random variables
- Write $\mathbf{E}[X_i] = \mu$ and $\mathbf{Var}(X_i) = \sigma^2$
- Set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mathbf{P}\left(a < \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) < b\right) = \Phi(b) - \Phi(a)$$

Interpretation of CLT

As $n \to \infty$

$$\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right) \approx Z, \quad Z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \quad \bar{X}_n - \mu \simeq \frac{\sigma Z}{\sqrt{n}}$$

$$\Rightarrow \quad \bar{X}_n \simeq \mu + \frac{\sigma Z}{\sqrt{n}}$$

$$\boxed{\bar{X}_n = E[X_i] + \text{Gauss. fluctuations of order } \frac{1}{\sqrt{n}}}$$

Problem 8.5 (1)

Situation: We have

- Fifty numbers rounded off to the nearest integer and then summed
- The individual round-off errors are uniformly distributed over $(-0.5, 0.5)$

Question:
Approximate the probability that the resultant sum differs from the exact sum by more than 3.

Model For $i = 1, \dots, n$, $n = 50$ let

X_i = i-th error

Hyp X_i's are iid, $X_i \sim \mathcal{U}([-0.5, 0.5])$

Recall: for $Y \sim \mathcal{U}([a,b])$ then

$$f_Y(y) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(y)$$

$$\mathbb{E}[Y] = \frac{a+b}{2} \qquad V(Y) = \frac{(b-a)^2}{12}$$

Here

$$\boxed{\begin{aligned} \mathbb{E}[X_i] &= 0 \qquad V(Y) = \frac{1}{12} = \sigma^2 \\ &= \mu \end{aligned}}$$

$n = 50$ $\quad \mu = 0$

$\sigma = \frac{1}{\sqrt{12}}$

We wish to compute

$$\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right)$$

in CLT: $\sqrt{n}\,\frac{(\bar{X}_n - \mu)}{\sigma}$

$$= 1 - \mathbb{P}\left(-3 \leq \sum_{i=1}^{n} X_i \leq 3\right)$$

$$= 1 - \mathbb{P}\left(-3/n \leq \bar{X}_n \leq 3/n\right)$$

$$= 1 - \mathbb{P}\left(\frac{(-3/n - \mu)}{\sigma}\sqrt{n} \leq \sqrt{n}\frac{(\bar{X}_n - \mu)}{\sigma} \leq \frac{(3/n - \mu)}{\sigma}\sqrt{n}\right)$$

CLT

$$\simeq 1 - \mathbb{P}\left(\frac{-3}{\sqrt{n}\,\sigma} \leq Z \leq \frac{3}{\sqrt{n}\,\sigma}\right)$$

$$= 1 - \mathbb{P}\left(-\frac{3\sqrt{12}}{\sqrt{50}} \leq Z \leq \frac{3\sqrt{12}}{\sqrt{50}}\right)$$

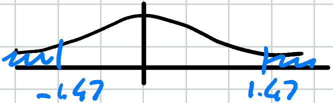
$$= 1 - \mathbb{P}\left(-1.47 \leq Z \leq 1.47\right)$$

Summary : We have found

$$\boxed{\mathbb{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right)}$$

$$\simeq 1-\mathbb{P}(-1.47 \le Z \le 1.47)$$

$$= \mathbb{P}(|Z| > 1.47)$$

$=$ 

$$= 2\,(1-\Phi(1.47))$$

$$\boxed{= \quad 0.14}$$

Problem 8.5 (2)

Model: Set

$$X_i \equiv i\text{-th error}$$

We wish to find

$$\mathbf{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right)$$

Law of X_i: We have

- X_i's i.i.d
- $X_i \sim \mathcal{U}([-0.5, 0.5)$
- $\mathbf{E}[X_i] = 0$, and $\mathbf{Var}(X_i) = \sigma^2 = \frac{1}{12}$

Problem 8.5 (3)

Application of CLT: Write

$$
\begin{aligned}
\mathbf{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right) &= \mathbf{P}\left(\left|\sqrt{n}\,\bar{X}_n\right| > \frac{3}{\sqrt{50}}\right) \\
&= \mathbf{P}\left(\left|\sqrt{n}\,\frac{\bar{X}_n}{\sigma}\right| > \frac{3\sqrt{12}}{\sqrt{50}}\right) \\
&\overset{CLT}{\simeq} \mathbf{P}\left(|Z| > 1.47\right) \\
&= .14
\end{aligned}
$$