# Outline

# Definition of probability

Sample space $S$

Example ③ : for $E, F$ with $E \cap F = \emptyset$, $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

**Definition 3.**

A probability is an application which assigns a number (chances to occur) to any event $E$. It must satisfy 3 axioms

1. $$0 \leq \mathbf{P}(E) \leq 1$$

2. "Probability that something happens" = $$\mathbf{P}(S) = 1$$

3. If $E_i E_j = \emptyset$ for $i, j \geq 1$ such that $i \neq j$, then
$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbf{P}(E_i)$$

# Easy consequence of the axioms

**Proposition 4.**

Let **P** be a probability on $S$. Then

1. "Probability that nothing happens" $= \mathbf{P}(\emptyset) = 0$
2. For $n \geq 1$,
   if $E_i E_j = \emptyset$ for $1 \leq i, j \leq n$ such that $i \neq j$ then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbf{P}(E_i)$$

# Example: dice tossing

Hyp: Dice is fair

Experiment: tossing one dice

Model: $S = \{1, \ldots, 6\}$ and $\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6}$

Def of probab: $\mathbf{P}(\{s\}) = \frac{1}{6}$, for all $s \in S$

Definition of an event We set

$$E = \text{"even number obtained"}$$

$\Rightarrow \mathbb{P}(E) = \frac{1}{2}$

$S = \{1, \dots, 6\}$ , $\mathbb{P}(\{i\}) = \frac{1}{6}$

# Dice tossing example

$E =$ "outcome is even number" $= \{2, 4, 6\}$

Thus

$$\boxed{\mathbb{P}(E)} = \mathbb{P}(\{2,4,6\}) =$$

pairwise disjoint

$$= \mathbb{P}(\{2\} \cup \{4\} \cup \{6\})$$

Prop4 - ②

$$= \mathbb{P}(\{2\}) + \mathbb{P}(\{4\}) + \mathbb{P}(\{6\})$$

$$= 3 \times \frac{1}{6} \boxed{= \frac{1}{2}}$$

# Probability of an event for dice tossing

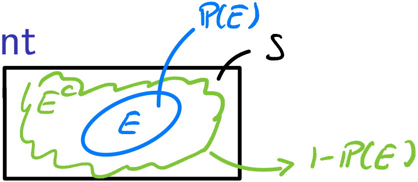
Computing the probability:

If $E$ = "even number obtained", then

$$\begin{aligned}\mathbf{P}(E) &= \mathbf{P}(\{2,4,6\}) = \mathbf{P}\left(\{2\} \cup \{4\} \cup \{6\}\right) \\ &= \mathbf{P}\left(\{2\}\right) + \mathbf{P}\left(\{4\}\right) + \mathbf{P}\left(\{6\}\right) = \frac{3}{6} \\ &= \frac{1}{2}\end{aligned}$$

# Outline

# Probability of a complement




**Proposition 5.**

Let

- $\mathbf{P}$ a probability on a sample space $S$
- $E$ an event

Then

$$\mathbf{P}\left(E^c\right) = 1 - \mathbf{P}(E)$$

# Proof

Use Axioms 2 and 3:

$$1 = \mathbf{P}(S) = \mathbf{P}\left(E \cup E^c\right) = \mathbf{P}\left(E\right) + \mathbf{P}\left(E^c\right)$$

# Probability of a subset


**Proposition 6.**

Let

- $\mathbf{P}$ a probability on a sample space $S$
- $E, F$ two events, such that $E \subset F$

Then

$$\mathbf{P}(E) \leq \mathbf{P}(F)$$

# Proof

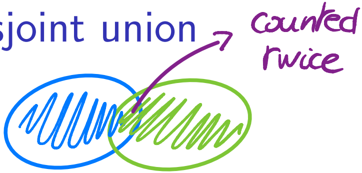
Decomposition of $F$: Write

$$F = E \cup E^c F$$

Use Axioms 1 and 3: Since $E$ and $E^c F$ are disjoint,

$$\mathbf{P}(F) = \mathbf{P}\left(E \cup E^c F\right) = \mathbf{P}\left(E\right) + \mathbf{P}\left(E^c F\right) \geq \mathbf{P}(E)$$

# Probability of a non disjoint union




**Proposition 7.**

Let

- $\mathbf{P}$ a probability on a sample space $S$
- $E, F$ two events , $E, F$ non necessarily disjoint

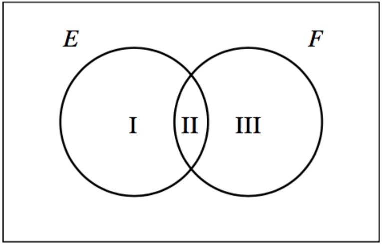
Then

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

# Proof

Decomposition of $E \cup F$:

$$E \cup F = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$$



# Proof (2)

Decomposition for probabilities: We have

$$\begin{aligned} \mathbf{P}(E \cup F) &= \mathbf{P}(\mathrm{I}) + \mathbf{P}(\mathrm{II}) + \mathbf{P}(\mathrm{III}) \\ \mathbf{P}(E) &= \mathbf{P}(\mathrm{I}) + \mathbf{P}(\mathrm{II}) \\ \mathbf{P}(F) &= \mathbf{P}(\mathrm{II}) + \mathbf{P}(\mathrm{III}) \end{aligned}$$

Conclusion: Since $\mathrm{II} = E \cap F$, we get

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(\mathrm{II}) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

# Application of Propositions 5 and 7

Experiment: dice tossing
$\hookrightarrow S = \{1, \ldots, 6\}$ and $\mathbf{P}(\{s\}) = \frac{1}{6}$ for all $s \in S$

Events: We consider the 2 events

$$A = \text{"even outcome"}$$
$$B = \text{"outcome multiple of 3"}$$

# Dice tossing example

A = "even outcome" = $\{2,4,6\}$

Aim: check that $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$A^c$ = "odd outcome" = $\{1,3,5\}$

Thus

$$\boxed{\mathbb{P}(A^c)} = \mathbb{P}(\{1\} \cup \{3\} \cup \{5\})$$

$$= \mathbb{P}(\{1\}) + \mathbb{P}(\{3\}) + \mathbb{P}(\{5\})$$

$$= \frac{3}{6} = \frac{1}{2} = \boxed{1 - \mathbb{P}(A)}$$

↗ Prop 5 verified

## Checking $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$A =$ "even outcome" $= \{2, 4, 6\}$

$B =$ "outcome mult. of 3" $= \{3, 6\}$

$$\mathbb{P}(B) = \mathbb{P}(\{3\}) + \mathbb{P}(\{6\}) = \frac{2}{6} = \frac{1}{3}$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(\{2, 3, 4, 6\}) = \frac{4}{6} = \frac{2}{3}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{6\}) = \frac{1}{6}$$

$$\Rightarrow \mathbb{P}(A \cup B) = 2/3 = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

We have checked Prop 7

# Application of Propositions 5 and 7 (Ctd)

Experiment: dice tossing
$\hookrightarrow S = \{1, \ldots, 6\}$ and $\mathbf{P}(\{s\}) = \frac{1}{6}$ for all $s \in S$

Events:
We consider $A =$ "even outcome" and $B =$ "outcome multiple of 3"
$\Rightarrow A = \{2, 4, 6\}$ and $B = \{3, 6\}$
$\Rightarrow \mathbf{P}(A) = 1/2$ and $\mathbf{P}(B) = 1/3$

Applying Propositions 5 and 7:
$\mathbf{P}(A^c) = 1 - \mathbf{P}(A) = 1/2$
$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1/2 + 1/3 - \mathbf{P}(\{6\}) = 2/3$

Verification:
$A^c = \{1, 3, 5\} \Rightarrow \mathbf{P}(A^c) = 1/2$
$A \cup B = \{2, 3, 4, 6\} \Rightarrow \mathbf{P}(A \cup B) = 4/6 = 2/3$

# Inclusion-exclusion identity

$n=2: \mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$

**Proposition 8.**

Let

- $\mathbf{P}$ a probability on a sample space $S$
- $n$ events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < \cdots < i_r \le n} \mathbf{P}\left(E_{i_1} \cdots E_{i_r}\right)$$

# Proof for $n = 3$

Apply Proposition 7:

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup E_3) &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}((E_1 \cup E_2)E_3) \\ &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}(E_1 E_3 \cup E_2 E_3)\end{aligned}$$

Apply Proposition 7 to $E_1 \cup E_2$ and $E_1 E_3 \cup E_2 E_3$:

$$\mathbf{P}(E_1 \cup E_2 \cup E_3) = \sum_{1 \leq i_1 \leq 3} \mathbf{P}(E_{i_1}) - \sum_{1 \leq i_1 < i_2 \leq 3} \mathbf{P}(E_{i_1} E_{i_2}) + \mathbf{P}(E_1 E_2 E_3)$$

Case of general $n$: By induction