Probability: On a sample space S, $\mathbb{P}(E)$ is defined for every event $E \subset S$ and

- $0 \leq \mathbb{P}(E) \leq 1$
- $\mathbb{P}(S) = 1$, $\mathbb{P}(\emptyset) = 0$
- If $E_1 \cap E_2 = \emptyset$, then
  $\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2)$

  → generalizations to $E_1, \dots, E_n$

Two properties

$$\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$$

$$\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2)$$

(inclusion-exclusion formula for $n = 2$)

Inclusion-exclusion for $n = 3$

$$\mathbb{P}(E_1 \cup E_2 \cup E_3)$$

$$= \quad \mathbb{P}(E_1) + \mathbb{P}(E_2) + \mathbb{P}(E_3) \qquad r = 1$$

$$- \{ \mathbb{P}(E_1 \cap E_2) + \mathbb{P}(E_1 \cap E_3) + \mathbb{P}(E_2 \cap E_3) \} \qquad r = 2$$

$$+ \quad \mathbb{P}(E_1 \cap E_2 \cap E_3) \qquad r = 3$$

Proof for $n = 3$

Apply Proposition 7:

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup E_3) &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}((E_1 \cup E_2)E_3) \\ &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}(E_1 E_3 \cup E_2 E_3)\end{aligned}$$

Apply Proposition 7 to $E_1 \cup E_2$ and $E_1 E_3 \cup E_2 E_3$:

$$\mathbf{P}(E_1 \cup E_2 \cup E_3) = \sum_{1 \le i_1 \le 3} \mathbf{P}(E_{i_1}) - \sum_{1 \le i_1 < i_2 \le 3} \mathbf{P}(E_{i_1} E_{i_2}) + \mathbf{P}(E_1 E_2 E_3)$$

Case of general n: By induction

Bounds for $\mathbf{P}(\cup_{i=1}^{n} E_i)$

Proposition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{1\leq i\leq n} \mathbf{P}(E_i)$$

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{1\leq i\leq n} \mathbf{P}(E_i) - \sum_{1\leq i_1<i_2\leq n} \mathbf{P}(E_{i_1} E_{i_2})$$

Bounds for $\mathbf{P}(\cup_{i=1}^n E_i)$ – Ctd

Proposition 10.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^n E_i\right)$$
$$\leq \sum_{1\leq i\leq n} \mathbf{P}(E_i) - \sum_{1\leq i_1<i_2\leq n} \mathbf{P}(E_{i_1} E_{i_2}) + \sum_{1\leq i_1<i_2<i_3\leq n} \mathbf{P}(E_{i_1} E_{i_2} E_{i_3})$$

Proof

Notation: Set

$$B_i = E_1^c \cdots E_{i-1}^c$$

Identity:

$$\mathbf{P}\left(\cup_{i=1}^n E_i\right) = \mathbf{P}(E_1) + \sum_{i=2}^n \mathbf{P}\left(B_i E_i\right)$$

Second identity: Since $B_i = (\cup_{j<i} E_j)^c$,

$$\mathbf{P}\left(B_i E_i\right) = \mathbf{P}\left(E_i\right) - \mathbf{P}\left(\cup_{j<i} E_j E_i\right)$$

Partial conclusion:

$$\mathbf{P}\left(\cup_{i=1}^n E_i\right) = \sum_{1\leq i\leq n} \mathbf{P}(E_i) - \sum_{1\leq i\leq n} \mathbf{P}\left(\cup_{j<i} E_j E_i\right)$$

Proof (2)

Recall:

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) = \sum_{1\leq i\leq n} \mathbf{P}(E_i) - \sum_{1\leq i\leq n} \mathbf{P}\left(\cup_{j<i} E_j E_i\right) \tag{1}$$

Direct consequence of (1):

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) \leq \sum_{1\leq i\leq n} \mathbf{P}(E_i) \tag{2}$$

Application of (2) to $\mathbf{P}(\cup_{j<i} E_j E_i)$:

$$\mathbf{P}\left(\cup_{j<i} E_j E_i\right) \leq \sum_{j<i} \mathbf{P}\left(E_j E_i\right)$$

Plugging into (1) we get

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) \geq \sum_{1\leq i\leq n} \mathbf{P}(E_i) - \sum_{j<i} \mathbf{P}\left(E_j E_i\right)$$

Outline

1. Introduction
2. Sample space and events
3. Axioms of probability
4. Some simple propositions
5. Sample spaces having equally likely outcomes
6. Probability as a continuous set function

Model

Hypothesis for this section: We assume

- $S = \{s_1, \ldots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \leq i \leq N$

Alert:

This is an important but
very particular case of probability space

Example: Toss 1 dice

$S = \{1, \dots, 6\} \rightarrow |S| = 6$

Hyp Dice is fair. Thus

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \dots = \mathbb{P}(\{6\})$$

$$= \frac{1}{6}$$

$$= \frac{1}{|S|}$$

Example 2: Toss 4 dice

$S = \{1, \dots, 6\}^4$, $|S| = 6^4 = 1296$

Hyp Dice are fair. Thus

$\mathbb{P}(\{1,1,1,1\}) = \mathbb{P}(\{1,1,1,2\})$

$= \dots = \mathbb{P}(\{6,6,6,6\}) = \frac{1}{1296}$

Example of uniform probability

Experiment: tossing 4 fair dice

Corresponding probability We take

- $S = \{1, \dots, 6\}^4$
- Probability defined by

$$\begin{aligned} \mathbf{P}(\{(1,1,1,1)\}) &= \mathbf{P}(\{(1,1,1,2)\}) = \cdots = \mathbf{P}(\{(6,6,6,6)\}) \\ &= \frac{1}{6^4} = \frac{1}{1296} \end{aligned}$$

Computing probabilities

Proposition 11.

Hypothesis: We assume

- $S = \{s_1, \ldots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \leq i \leq N$

In this situation, let $E \subset S$ be an event. Then

$$\mathbf{P}(E) = \frac{\mathrm{Card}(E)}{N} = \frac{|E|}{N} = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$$

Example 1: Tossing 1 dice with

$$\mathbb{P}(\{1\}) = \dots = \mathbb{P}(\{6\}) = \frac{1}{6}$$

$E =$ "even outcome" $= \{2, 4, 6\}$

Since the probability is uniform,

$$\mathbb{P}(E) \overset{\text{Prop 11}}{=} \frac{|E|}{|S|} = \frac{3}{6} = \frac{1}{2}$$

Example: tossing one dice

Model: tossing one dice, that is

$$S = \{1, \dots, 6\}, \qquad \mathbf{P}(\{s_i\}) = \frac{1}{6}$$

Computing a simple probability: Let $E =$ "even outcome". Then

$$\mathbf{P}(E) = \frac{|E|}{N} = \frac{3}{6} = \frac{1}{2}$$

Main problem: compute $|E|$ in more complex situations
$\hookrightarrow$ Counting

Example: drawing balls (1)

Situation: We have

- A bowl with 6 White and 5 Black balls
- We draw 3 balls

Problem: Compute

$$\mathbf{P}(E), \quad \text{with} \quad E = \text{"Draw 1 W and 2 B"}$$

Experiment: 6 W, 5 B, we draw 3 balls

S = " all combinations of size 3 of 6 W, 5B"

Balls: $\overbrace{1, \dots, 6}^{W}, \overbrace{7, \dots, 11}^{B}$. Thus

$$S = \{ (i_1, i_2, i_3) \in \{1, \dots, 11\}^3 ; \; i_1 \neq i_2, \; i_2 \neq i_3, \; i_3 \neq i_1 \}$$

Note: here $(1,2,6) \neq (2,1,6)$, the order matters

E = " 2 B, 1 W "

= BBW ∪ BWB ∪ WBB ← order matters

Description of BBW

$$BBW = \{ (i_1, i_2, i_3) \in \{1, .., 11\}^3 ; \begin{matrix} i_1 \neq i_2, i_2 \neq i_3 \\ i_3 \neq i_1 \end{matrix}$$
$$7 \leq i_1 \leq 11, \quad 7 \leq i_2 \leq 11, \quad 1 \leq i_3 \leq 6 \}$$

Hyp: Outcomes are equally likely.
Thus $\forall (i_1, i_2, i_3) \in S$

$$\mathbb{P}(\{(i_1, i_2, i_3)\}) = \frac{1}{|S|} =$$

$$= \frac{1}{11 \cdot 10 \cdot 9} = \frac{1}{990}$$

We get

$$\mathbb{P}(BBW) \overset{\text{Prop11}}{=} \frac{|BBW|}{|S|} = \frac{5\times 4\times 6}{990}$$

$$= \mathbb{P}(BWB) = \mathbb{P}(WBB)$$

Thus

disjoint sets

$$\boxed{\mathbb{P}(E)} = \mathbb{P}(BBW)+\mathbb{P}(BWB)+\mathbb{P}(WBB)$$

$$= 3\times \frac{5\times 4\times 6}{990} \quad \boxed{\simeq 36.4\%}$$

Rmk : Here we had a model for which the order mattered.

We have another possible model, for which the order does not matter

→ combination numbers

Example: drawing balls (2)

Model 1: We take

- $S = \{$Ordered triples of balls, tagged from 1 to 11$\}$
- $\mathbf{P}$ = Uniform probability on S

Computing $|S|$: We have

$$|S| = 11 \cdot 10 \cdot 9 = 990$$

Decomposition of E: We have

$$E = \mathrm{WBB} \cup \mathrm{BWB} \cup \mathrm{BBW}$$

Example: drawing balls (3)

Counting E:

$$|E| = |\mathrm{WBB}| + |\mathrm{BWB}| + |\mathrm{BBW}| = 3 \times (6 \times 5 \times 4) = 360$$

Probability of E: We get

$$\mathbf{P}(E) = \frac{|E|}{|S|} = \frac{360}{990} = \frac{4}{11} = 36.4\%$$