Outline

1 Introduction

2 Conditional probabilities

3 Bayes's formula

4 Independent events

5 Conditional probability as a probability

Global objective

Aim: Introduce conditional probability, whose interest is twofold

1. Quantify the effect of a prior information on probabilities
2. If no prior information is available, then independence
   $\hookrightarrow$ simplification in probability computations

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1. Introduction
2. Conditional probabilities
3. Bayes's formula
4. Independent events
5. Conditional probability as a probability

Example of conditioning

Dice tossing: We consider the following situation

- We throw 2 dice
- We look for $\mathbf{P}(\text{sum of 2 faces is 9})$

Without prior information:

$$\mathbf{P}\left(\text{sum of 2 faces is 9}\right) = \frac{1}{9}$$

Changes with additional information:

If we know that first face is $= 4$, then

$\hookrightarrow$ how does it affect $\mathbf{P}\left(\text{sum of 2 faces is 9}\right)$?

$\mathbb{P}(\text{sum} = 9)$ without prior information

$$S = \{(i,j) ; \; i,j \in \{1, \dots, 6\}\}$$

$$= \{1, \dots, 6\}^2$$

$$\mathbb{P}(\{(i,j)\}) = \frac{1}{36}$$

$$\text{"Sum} = 9\text{"} = \{(4,5); (5,4); (3,6); (6,3)\}$$

$$\Rightarrow \mathbb{P}(\text{Sum} = 9) = \frac{4}{36}$$

$$\boxed{\mathbb{P}(\text{Sum} = 9) = \frac{1}{9}}$$

Prior information: First dice is a 4

Will it change $\mathbb{P}(\text{Sum} = 9)$?

We get a reduced sample space

$S' = \{ (4,1); (4,2); \dots ; (4,6) \}$

Within S', we have

"Sum = 9" $= \{ (4,5) \}$

The probability becomes

$$p = \frac{1}{6} \quad \left(\neq \frac{1}{9} \right)$$

Example of conditioning

Probability with additional information: If first face is $= 4$, then

- Only 6 possible results:

$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$

- Among them, only $(4,5)$ gives sum $= 9$
- Probability of having sum $= 9$ becomes

$$p = \frac{1}{6}$$

Conclusion:

We need to formalize this type of computation

General definition

Definition 1.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events, such that $\mathbf{P}(F) > 0$

probab of E given F

Then

$$\mathbf{P}(E|\,F) = \frac{\mathbf{P}(E\,F)}{\mathbf{P}(F)}$$

Example: examination (1)

Situation:
Student taking a one hour exam

Hypothesis: For $x \in [0, 1]$ we have

$$\mathbf{P}(L_x) = \frac{x}{2}, \tag{1}$$

where the event L_x is defined by

$$L_x = \{\text{student finishes the exam in less than } x \text{ hour}\}$$

Question: Given that the student is still working after .75h
$\hookrightarrow$ Find probability that the full hour is used

Student exam example

$S = \{$ amount of time the student needs $\}$

$= [0,1] \cup \{1^+\}$ → student needs more than 1h

Probab: specified by the values

$\mathbb{P}($ student needs less than $x)$

$= \mathbb{P}(L_x) = \frac{x}{2}$ → not a uniform probability

In particular, $\mathbb{P}(L_1) = \frac{1}{2}$

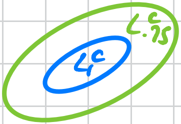
$$\Rightarrow \boxed{\mathbb{P}(\{1^+\}) = \mathbb{P}(L_1^c) = \frac{1}{2}}$$

Aim: compute

P(student need more than 1h | student needs more than .75)

$$= \boxed{\mathbb{P}(L_1^c \mid L_{.75}^c)} \overset{?}{\gtreqless} \mathbb{P}(L_1^c)$$

$$\overset{\text{def}}{=} \frac{\mathbb{P}(L_1^c \cap L_{.75}^c)}{\mathbb{P}(L_{.75}^c)}$$



$$= \frac{\mathbb{P}(L_1^c)}{\mathbb{P}(L_{.75}^c)} = \frac{1 - \mathbb{P}(L_1)}{1 - \mathbb{P}(L_{.75})} = \frac{1 - 1/2}{1 - 3/8}$$

$$\boxed{= \frac{4}{5}} > \frac{1}{2} = \mathbb{P}(L_1^c)$$

Example: examination (2)

Model: We wish to find

$$\mathbf{P}\left(L_1^c \mid L_{.75}^c\right)$$

Computation: We have

$$\begin{aligned}
\mathbf{P}\left(L_1^c \mid L_{.75}^c\right) &= \frac{\mathbf{P}\left(L_1^c L_{.75}^c\right)}{\mathbf{P}\left(L_{.75}^c\right)} \\
&= \frac{\mathbf{P}\left(L_1^c\right)}{\mathbf{P}\left(L_{.75}^c\right)} \\
&= \frac{1-\mathbf{P}\left(L_1\right)}{1-\mathbf{P}\left(L_{.75}\right)}
\end{aligned}$$

Conclusion: Applying (1) we get

$$\mathbf{P}\left(L_1^c \mid L_{.75}^c\right) = .8$$