# Simplification for uniform probabilities

General situation: We assume

- $S = \{s_1, \ldots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \leq i \leq N$

Alert:
This is an important but very particular case of probability space

Conditional probabilities in this case:
Reduced sample space, i.e

Conditional on $F$, all outcomes in $F$ are equally likely

$\mathbb{P}(E|F)$

# Reduced sample space - 1 dice roll

$S = \{1, \dots, 6\}$ , $\mathbb{P}(\{i\}) = \frac{1}{6}$

$F =$ " even outcome " $= \{2,4,6\}$

$E =$ " outcome is 4 " $= \{4\}$ $\left(\Rightarrow \mathbb{P}(E) = \frac{1}{6}\right)$

If we condition on F, the reduced sample space becomes

$\boxed{\tilde{S} = \{2, 4, 6\} \text{ , with every outcome having prob. } = \frac{1}{3}}$

According to reduced sample space, since $E = \{4\} \subset F = \{2,4,6\}$, we have

$$\boxed{\mathbb{P}(E \mid F) = \frac{1}{3}}$$

Rmk we could have computed

$$\boxed{\mathbb{P}(E \mid F)} = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(\{4\} \cap \{2,4,6\})}{\mathbb{P}(\{2,4,6\})}$$

$$= \frac{\mathbb{P}(\{4\})}{\mathbb{P}(\{2,4,6\})} = \frac{1/6}{1/2} \boxed{= \frac{1}{3}}$$

reduced sample space simpler

# Example: family distribution (1)

Intuition: conditional probability is not 0.5 !

**Situation:**

The Popescu family has 10 kids

**Questions:**

1. If we know that 9 kids are girls
   $\hookrightarrow$ find the probability that all 10 kids are girls

# Example: family distribution (1)

Situation:

The Popescu family has 10 kids

Questions:

1. If we know that 9 kids are girls
   $\hookrightarrow$ find the probability that all 10 kids are girls
2. If we know that the first 9 kids are girls
   $\hookrightarrow$ find the probability that all 10 kids are girls

G B

# Family of 10 kids example

$$S = \{ (i_1, \dots, i_{10}) \; ; \; i_j \in \{G, B\} \}$$

$$= \{G, B\}^{10}$$

realistic ?

**Hyp** : every outcome equally likely

$$\mathbb{P}(\{(i_1, \dots, i_{10})\}) = \frac{1}{2^{10}} = \frac{1}{1024}$$

$S = \{G, B\}^{10}$

# First conditioning

$F_1$ = "at least 9 kids are girls"

$$= \{(G, G, G, \ldots, G); (G, \ldots, G, B); (G, \ldots, G, B, G); \ldots (B, G, \ldots, G)\} \Rightarrow |F_1| = 11$$

We get

reduced sample space

$$\boxed{\mathbb{P}(\{(G, \ldots, G)\} \mid F_1)} = \frac{1}{|F_1|} = \boxed{\frac{1}{11}} \neq \frac{1}{2}!$$

# Second conditioning

$F_2$ = " First nine kids are girls "

$= \{ (G, \dots, G, G) \, ; \, (G, \dots, G, B) \}$

Thus

$\boxed{\mathbb{P}( \{ (G, \dots, G) \} \mid F_2 \}}$

reduced sample space

$= \frac{1}{|F_2|} \boxed{= \frac{1}{2}}$

# Example: family distribution (2)

Model:

- $S = \{G, B\}^{10}$
- Uniform probability: for all $s \in S$,

$$\mathbf{P}(\{s\}) = \frac{1}{2^{10}} = \frac{1}{1024}$$

# Example: family distribution (3)

First conditioning: We take

$$F_1 = \{(G, \ldots, G); \,(G, \ldots, G, B); \,(G, \ldots, G, B, G); \cdots ; \,(B, G, \ldots, G)\}$$

Reduced sample space:
Each outcome in $F_1$ has probability $\frac{1}{11}$

Conditional probability:

$$\mathbf{P}\left(\{(G, \ldots, G)\} \middle| \, F_1\right) = \frac{1}{11}$$

# Example: family distribution (4)

Second conditioning: We take

$$F_2 = \{(G, \ldots, G); \, (G, \ldots, G, B)\}$$

Reduced sample space:
Each outcome in $F_2$ has probability $\frac{1}{2}$

Conditional probability:

$$\mathbf{P}\left(\{(G, \ldots, G)\} \,|\, F_2\right) = \frac{1}{2}$$

# Example: bridge game (1)

Bridge game:

- 4 players, E, W, N, S
- 52 cards dealt out equally to players → 13 cards each

Conditioning: We condition on the set

$$F = \{N + S \text{ have a total of 8 spades}\}$$

Question: Conditioned on $F$,
Probability that E has 3 of the remaining 5 spades

## Bridge example, sample space

S = "Assignements of 52 cards into 13 + 13 + 13 + 13"

$$\Rightarrow |S| = \binom{52}{13\ 13\ 13\ 13} \simeq 5 \times 10^{28}$$

We condition on

$F = \{N+S \text{ have } 8 \text{ spades}\}$ = "Assignments of 26 cards into 13+13, with 5 ♤"

$$\Rightarrow \boxed{|F| = \binom{26}{13}}$$

We wish to compute

$\mathbb{P}($ E has $=3$ of the remaining 5 ♤ | F $)$

reduced sample space

$$= \frac{\binom{5}{3}\binom{26-5}{10}}{\binom{26}{13}} \approx .34$$

# Example: bridge game (2)

Model: We take

$$S = \{\text{Divisions of 52 cards in 4 groups}\}$$

and we have

- Uniform probability on $S$
- $|S| = \binom{52}{13,13,13,13} \simeq 5.36\ 10^{28}$

Reduced sample space: Conditioned on $F$,

$$\tilde{S} = \{\text{Combinations of 13 cards among 26 cards with 5 spades}\}$$

and $|\tilde{S}| = 10,400,600$

# Example: bridge game (3)

Conditional probability:

$$\mathbf{P}\left(\text{E has 3 of the remaining 5 spades} \mid F\right) = \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \simeq .339$$

# Intersection and conditioning

Situation:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Question: Let

- $R_1$ = 1st ball drawn is red
- $R_2$ = 2nd ball drawn is red

Then find $\mathbf{P}(R_1 R_2)$

12 balls, 8 R, 4 W

Recall:

$$\mathbb{P}(R_2 \mid R_1) = \frac{\mathbb{P}(R_2 \cap R_1)}{\mathbb{P}(R_1)}$$

$$\Rightarrow \boxed{\mathbb{P}(R_2 \cap R_1) = \mathbb{P}(R_2 \mid R_1)\,\mathbb{P}(R_1)}$$

Here

$$\boxed{\mathbb{P}(R_2 \cap R_1)} = \mathbb{P}(R_1)\,\mathbb{P}(R_2 \mid R_1)$$

$$= \frac{8}{12} \times \frac{7}{11} \boxed{\simeq .42}$$

very simple computation

# Intersection and conditioning (2)

Recall:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Computation: We have

$$\mathbf{P}(R_1 R_2) = \mathbf{P}(R_1)\mathbf{P}(R_2 |\, R_1)$$

Thus

$$\mathbf{P}(R_1 R_2) = \frac{8}{12}\,\frac{7}{11} = \frac{14}{33} \simeq .42$$