MA/STAT 416 Spring 2020
Probability

Midterm Exam

- You can use a calculator.
- A 2 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 60 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

Name:

Problem 1. Let $ABCD$ be a square with unit area. Let $\alpha, \beta, \gamma, \delta$ be random points distributed uniformly on $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ respectively. Let S be the area of the quadrangle $\alpha\beta\gamma\delta$. Find $\mathbf{E}[S]$. Hint: you may use $S = \frac{1}{2}|\vec{\alpha\gamma} \times \vec{\beta\delta}| = \frac{1}{2}|\det(\vec{\alpha\gamma}, \vec{\beta\delta})|$ if you want.

Problem 2. In the Boilermaker dinning hall, the weight X of the sirloin steak follows an exponential distribution with mean 12 oz. If the steak if less than 15 oz, it is already included in your meal plan, which means that you do not pay extra money. If it is more than 15 oz, you need to pay an extra amount equal to $X - 15$ dollars. Let Y be the amount charged on top of your meal plan.

2.1. Find the cdf of the random variable Y. In addition, plot the cdf.

2.2. Find the probability that you are charged extra money for the second time when you get your 5th serving.

Problem 3. The number of customers coming to Frodo and Neo's ice cream shop in one hour is distributed as a Poisson random variable, whose average is 6 customers per hour. They open their shop at 11 am. If there are more than 2 (namely > 2) customers in the first 10 minutes after opening, they provide a free scoop to the third customer. They have a grand opening on April 1st and the shop is open every day.

3.1. Find the probability that the ice cream shop provides a free scoop on a typical business day.

3.2. What is the probability that the ice cream shop provides a free scoop for the third time on Apr 10?

3.3. What is the probability that the ice cream shop provides a free scoop more than 10 times in April (note: there are 30 days in April)?

Problem 4. Let X and Y be independent Poisson random variables with respective parameters λ_1 and λ_2. Determine the conditional distribution of X given that $X+Y = n$.

Problem 5. Consider two independent random variables (X_1, X_2) such that X_1 follows an exponential distribution with mean 2, and X_2 follows a uniform distribution on $[0, 2\pi]$. Let $Y_1 = \sqrt{X_1}\cos(X_2)$ and $Y_2 = \sqrt{X_1}\sin(X_2)$. Show that Y_1 and Y_2 are independent standard normal variables.