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MATH 416 PROBABILITY SPRING 2019
PROFESSOR WANG

Final Exam (10 problems, maximum 120 points +10 points extra credit)
Tuesday, April 30, 2019

You have 120 minutes to complete this exam. Books, notes, calculators, cell-phones and collaboration are not allowed

Unless otherwise stated, show all of your work; heroic simplification is unnecessary. Full credit may not be given for an answer alone, and partial credit may be given for facts relevant to the solution. Multiple answers for any problem earn **zero** credit.

Loose pages will be ignored. Circle your final answer.

Good luck!

1. For a sequence of i.id random variables $X_1, X_2, \dots$, $\mathbb{E}(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2$. Let $S_n = X_1 + \cdots + X_n$ then, when $n \to \infty$,

(a) (2 pts) Central limit theorem tells us that S_n asymptotically has $\boxed{}$ (type) distribution with parameter(s) $\boxed{}$.

In other words we know that in distribution (fill the boxes) (2pts)

$$\lim_{n\to\infty} \frac{S_n - \boxed{}}{\boxed{}} = \mathcal{N}(0,1).$$

(b) (1pts) Law of large numbers tells us that $\frac{S_n}{n}$ converges to $\boxed{}$ when $n \to \infty$.

(c) (5 pts) Given a random variable Y, the best use of Y to approximate another random variable X is (in terms a function of Y) $g(Y) = \boxed{}$.

By best approximation we mean $\boxed{}$ and $\boxed{}$. (Hint: about mean and variance of X and $g(Y)$.)

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2. Let X and Y be independent, identically distributed random variables whose density $f(x)$ is given by

$$f(x) = \begin{cases} ce^{-x} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(hint: pay attention that X, Y are not exponential random variables.)

(a) (5 pts) Find c

(b)(5 pts) Find the moment generating function $M_X(t)$ of X.

(c) (5 pts) Find the moment generating function $M_{X-Y}(t)$ of $X - Y$.

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3. The joint probability density function of X and Y is given by

$$f(x,y) = \begin{cases} c\,(x+y)\,, & 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) (5 pts) Find c.

(b) (5 pts) Find conditional density $f_{X|Y}(x|y)$ for $y \in (0,1)$. (hint: specify the region.)

(c) (5 pts) Find $\mathbb{P}(Y > 1/2 \mid X < 1/2)$.

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(d) (5 pts) Find $\mathbb{E}(X^2|Y=y)$ for $y \in (0,1)$.

4. (5 pts) A pond contains a random number $N \geq 30$ of fish, of which 30 are carp. We know that $\mathbb{E}(N) = 60$ and $\mathbb{E}(\frac{1}{N}) = \frac{1}{50}$. If 20 fish are caught (selected randomly), and X is the number of carp among the 20. Compute $\mathbb{E}(X)$. (Hint: write X as the sum of indicator random variables and using conditioning.

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5. (5 pts) Suppose that a university sells 225 permits for a student parking lot with 190 spots, and the students are independent and identically distributed, wanting to park in the lot 80% of the time. Use the central limit theorem to compute the probability that at least one student won't be able to park in the lot. (hint: use continuity correction).

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6. From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.

(a) (5pts) Give an upper bound for the probability that a student's test score will exceed 85.

(b) (5pts) Suppose, in addition, that the professor knows that the variance of a student's test score is equal to 25. What can be said about the probability that a student will score between 65 and 85?

(c) (5pts) How many students would have to take the examination to ensure, with probability at least 0.9, that the class average would be within 5 of 75? Do **NOT** use the central limit theorem.

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7. A Big apple has a random mass that's normally distributed with mean 160 (grams) and standard deviation 4. A Huge apple's mass is $\mathcal{N}(170; \mathrm{Var} = 36)$.

(a)(5pts) Find the probability that two Big apple and one Huge apple together have mass more than 500 grams.

(b)(Extra credit 5pts) Find the probability that a Big apple weighs 4.4 grams more than a Huge one. Give the explicit final answer and explain your answer. ($\sqrt{52} \approx 7.2$)

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8. Let X and Y be random variables with joint probability mass function defined by:

	$Y=-1$	$Y=0$	$Y=2$	$Y=6$
$X=-2$	$3/27$	$1/27$	$1/27$	$3/27$
$X=1$	$6/27$	0	$3/27$	$3/27$
$X=3$	0	0	$3/27$	$4/27$

(a) (5 pts) Compute $P(X<0, Y<0)$

(b) (5 pts) Compute conditional pmf of X given $Y=-1$.

(c) (5 pts) Compute $E[XY]$.

(d) (5 pts) Are X and Y independent? Explain.

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9. A fair coin is tossed twice independently. Let X and Y be the indicator random variables of the events that "H" appear in the 1st and 2nd toss respectively.

(a) (5pts) Compute the covariance of $X + Y$ and $X - Y$ (simplify).

(b) (5pts) Are $X + Y$ and $X - Y$ independent? Justify your answer clearly.

(c) (Extra credit 5pts) Find the moment generating function $M_{X+Y}(t)$ of $X + Y$ (hint: your answer should be a function of t and can contain unsimplified finite sums).

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10. Let X be a random variable with cumulative distribution function (cdf) given by

$$F_X(x) = \begin{cases} 0 & \text{if } -\infty < x < 0, \\ x/3 & \text{if } 0 \leq x < 1, \\ 1/2 & \text{if } 1 \leq x < 2, \\ x/4 & \text{if } 2 \leq x < 3, \\ 1 & \text{if } 3 \leq x. \end{cases}$$

(a) (5 pts) What should be the value of $\mathbb{P}(X = 1)$?

(b) (5 pts) Compute $\mathbb{P}(X < 3)$.

(c) (5 pts) Compute $\mathbb{P}(2 < X < 3)$.

(d) Compute $\mathbb{P}(2 < X \leq t \mid 2 < X < 3)$ for $t \in (2, 3)$.

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Area $\Phi(x)$ under the standard normal curve to the left of x

e.g. $\Phi(1.25) = 0.8944$ (6-th element in the row for 1.2), $\Phi(2.51) = 0.9940$.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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Name	**PMF or PDF**	**Mean**	**Variance**
Ber(p)	$\mathbb{P}(X=1)=p, \mathbb{P}(X=0)=1-p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$ for $k=0,1,\ldots,n$	np	$np(1-p)$
Geom(p)	$p(1-p)^{k-1}$ for $k=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\binom{k-1}{r-1}p^r q^{k-r}$ for $k=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
HGeom(N,m,n)	$\frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$ for $k=0,1,\ldots,n$	$\frac{nm}{N}$	$\frac{N-n}{N-1}\cdot\frac{nm(N-m)}{N^2}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^k}{k!}$ for $k=0,1,\ldots$	λ	λ
Uniform(a,b)	$\frac{1}{b-a}$ for $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ for $x\in(-\infty,\infty)$	μ	σ^2
Exp(λ)	$\lambda e^{-\lambda x}$ for $x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(a,λ)	$\frac{\lambda^a x^{a-1}e^{-\lambda x}}{\Gamma(a)}$ for $x>0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$

TABLE 1. **Table of Important Distributions.**