Midterm Fall 24 - Solutions

Problem 1. We throw a pair of dice until the sum of faces shows either 5 or 7.

1.1. Specify the state space S related to this experiment.

Here we have

$$S = \{\text{sequences of double throws in } \{1,\dots,6\}^2\}$$

Otherwise stated

$$\boxed{S = \left(\{1,\dots,6\}^2\right)^{\mathbb{N}_*}} ,$$

where

$$\mathbb{N}_* = \{1, 2, \dots\}$$

1.2. Let S_j be the random variable defined by $S_j =$ "Sum of the 2 faces for the j-th roll". Find the pmf of S_j.

For a given throw we have

$\mathbb{P}(\{(i_1, i_2)\}) = \frac{1}{36}, \quad \forall (i,j) \in \{1,..,6\}^2$

Then S_j takes the following values

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

We deduce that $S_j \in \{2, \dots, 12\}$ and the pmf $p(j)$ is given by

j	2	3	4	5	6	7	8	9	10	11	12
$p(j)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

1.3. We call A the event "5 occurs before 7". We also call A_n the event "5 occurs before 7, and this happens exactly on the n-th roll". Express A_n in terms of the random variables S_j for $j \leq n$.

A_n

$= (S_1 \notin \{5,7\}) \cap \dots \cap (S_{n-1} \notin \{5,7\}) \cap (S_n = 5)$

Otherwise stated:

$$\boxed{A_n = \left(\bigcap_{j=1}^{n-1} (S_j \notin \{5,7\}) \right) \cap (S_n = 5)}$$

1.4. Compute $\mathbf{P}(A)$ by using the relation $A = \cup_{n=1}^{\infty} A_n$.

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right)$$

$$= \sum_{n=1}^{\infty} \mathbb{P}(A_n) \qquad \text{(disjoint } A_n\text{'s)}$$

$$= \sum_{n=1}^{\infty} \mathbb{P}\left(\bigcap_{j=1}^{n-1} (S_j \notin \{5,7\}) \cap (S_n = 5)\right) \qquad \text{(cf. 1-3)}$$

$$= \sum_{n=1}^{\infty} \prod_{j=1}^{n-1} \mathbb{P}(S_j \notin \{5,7\}) \times \mathbb{P}(S_n = 5) \qquad (S_j\text{'s } \perp\!\!\!\perp)$$

$$= \sum_{n=1}^{\infty} \left(1 - \frac{10}{36}\right)^{n-1} \frac{4}{36} \qquad \text{(from pmf of } S_j\text{)}$$

$$= \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1}$$

$$= \frac{1}{9} \frac{1}{1 - \frac{13}{18}} \qquad \text{(sum geometric series)}$$

$$= \frac{1}{9} \frac{18}{5}$$

we get

$$\boxed{\mathbb{P}(A) = \frac{2}{5}}$$

1.5. We now define 3 events:

$$F_5 = \{S_1 = 5\}, \qquad F_7 = \{S_1 = 7\}, \qquad F_{\neq} = \{S_1 \notin \{5,7\}\}$$

Compute $\mathbf{P}(A)$ by conditioning on F_5, F_7 and $F_{\neq}$.

$$\mathbb{P}(A) = \mathbb{P}(A \mid F_5)\, \mathbb{P}(F_5) + \mathbb{P}(A \mid F_7)\, \mathbb{P}(F_7) + \mathbb{P}(A \mid F_{\neq})\, \mathbb{P}(F_{\neq}) \quad (1)$$

(Bayes 1)

$$= 1 \times \mathbb{P}(F_5) + 0 \times \mathbb{P}(F_7) + \mathbb{P}(A)\, \mathbb{P}(F_{\neq}),$$

where we justify $\mathbb{P}(A \mid F_{\neq}) = \mathbb{P}(A)$ by writing

$$\mathbb{P}(A \mid F_{\neq}) = \frac{\mathbb{P}\left(S_1 \notin \{5,7\} \cap \left(\bigcup_{n=2}^{\infty} \tilde{E}_n\right)\right)}{\mathbb{P}(S_1 \notin \{5,7\})},$$

and $\tilde{E}_n = (S_2 \notin \{5,7\}) \cap \dots \cap (S_{n-1} \notin \{5,7\}) \cap (S_n = 5)$.
Since the $S_2, \dots, S_{n-1}$ are $\perp\!\!\!\perp$ of S_1, we get

$$\mathbb{P}(A \mid F_{\neq}) = \frac{\mathbb{P}(S_1 \notin \{5,7\})\, \mathbb{P}\left(\bigcup_{n=2}^{\infty} \tilde{E}_n\right)}{\mathbb{P}(S_1 \notin \{5,7\})}$$

$$= \mathbb{P}\left(\bigcup_{n=2}^{\infty} \tilde{E}_n\right)$$

$$= \mathbb{P}(A)$$ ($\tilde{E}_n$ are shifted versions of E_n)

Going back to $\mathbb{P}(A)$: we have seen in (1) that

$$\mathbb{P}(A) = \mathbb{P}(F_5) + \mathbb{P}(F_7)\,\mathbb{P}(A)$$

$$= \mathbb{P}(S_1 = 5) + \mathbb{P}(S_1 \notin \{5,7\})\,\mathbb{P}(A)$$

$$= \frac{1}{9} + \frac{13}{18}\,\mathbb{P}(A)$$

Solving the linear equation we get

$$\left(1 - \frac{13}{18}\right)\mathbb{P}(A) = \frac{1}{9}$$

$$\frac{5}{18}\,\mathbb{P}(A) = \frac{1}{9}$$

$$\boxed{\mathbb{P}(A) = \frac{2}{5}}$$

Problem 2. In order to detect whether a suspect is lying, the police sometimes use polygraphs. Let $A = \{\text{polygraph indicates lying}\}$ and $B = \{\text{the suspect is lying}\}$. If the suspect is lying, there is a 88% chance of detecting it; if the suspect is telling the truth, 86% of time the polygraph will confirm it. We assume that 1% of the time the suspects lie.

2.1. If the result of the polygraph shows that the suspect is lying, what is the chance that this person is really lying?

Data we have

$\mathbb{P}(A \mid B) = .88$ $\mathbb{P}(B) = .01$

$\mathbb{P}(A^c \mid B^c) = .86$

Application of Bayes

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\,\mathbb{P}(B)}{\mathbb{P}(A \mid B)\,\mathbb{P}(B) + \mathbb{P}(A \mid B^c)\,\mathbb{P}(B^c)}$$

$$= \frac{.88 \times .01}{.88 \times .01 + .14 \times .99}$$

$$= \frac{88}{88 + 14 \times 99}$$

$$\boxed{\mathbb{P}(B \mid A) \simeq 6\%}$$

$\longrightarrow$ not accurate!

Problem 3. Two teams play a series of games that ends when one of them has won 3 games. Suppose that each game played is, independently, won by team A with probability $p \in (0,1)$. We call N the number of games played.

3.1. What is the state space for the random variable N?

State space

In order to see $i = 3$ wins, one needs to play

(i) A minimum of 3 games

(ii) A maximum of 5 games

Thus

$$N \in \{3, 4, 5\}$$

$$\boxed{\text{The state space is } \{3,4,5\}}$$

3.2. Compute the pmf for the random variable N.

pmf

The pmf is given by

$p(3) = \mathbb{P}(N=3)$, $p(4) = \mathbb{P}(N=4)$

$p(5) = \mathbb{P}(N=5)$

There is an underlying sample space

$$S = \bigcup_{j=3}^{5} \{0,1\}^j ,$$

where $1 \equiv$ team A wins. We have

(i) $(N=3) = \{ (0,0,0) ; (1,1,1) \}$

$\Rightarrow \quad \mathbb{P}(N=3) = p^3 + (1-p)^3$

(ii) $(N=4) =$

$\{ (0,0,1,0); (0,1,0,0); (1,0,0,0),$
$(1,1,0,1); (1,0,1,1); (0,1,1,1) \}$

$\Rightarrow \mathbb{P}(N=3) = 3\,p^3(1-p) + 3\,p(1-p)^3$

(iii) $(N=5) =$

$\{(0,0,1,1);(0,1,0,1);(0,1,1,0)$
$(1,0,0,1);(1,0,1,0);(1,1,0,0)\}$

$\Rightarrow P(N=5)$
$= 6\, p^2 (1-p)^2$

We have obtained

$$\boxed{\begin{aligned} P(3) &= p^3 + (1-p)^3 \\ P(4) &= 3p^3(1-p) + 3p(1-p)^3 \\ P(5) &= 6p^2(1-p)^2 \end{aligned}}$$

3.3. Compute $\mathbf{E}[N]$. *Note:* One can express the result as a function of $x = p(1-p)$.

Expected value We have

$$E[N] = \sum_{i=3}^{5} i\, p(i)$$

$$= 3\left(p^3 + (1-p)^3\right)$$

$$+ 12\left(p^3(1-p) + p(1-p)^3\right)$$

$$+ 30\, p^2(1-p)^2$$

$$= 3 - 3\left(3p^2(1-p) + 3p(1-p)^2\right)$$

$$+ 12\left(p^2(1-p)p + p(1-p)^2(1-p)\right)$$

$$+ 15\left(p^2(1-p)(1-p) + p(1-p)^2 p\right)$$

$$= 3$$

$$+ p^2(1-p)\{-9 + 12p + 15(1-p)\}$$

$$+ p(1-p)^2\{-9 + 12(1-p) + 15p\}$$

$$= 3 + p^2(1-p)\{3 + 3(1-p)\}$$

$$+ p(1-p)^2\{3 + 3p\}$$

We thus get

$$E[N] = 3\{1 + p(1-p)\left(p(1+(1-p)) + (1-p)(1+p)\right)\}$$

$$= 3\{1 + p(1-p)(1+2p(1-p))\}.$$

$E[N]$ as a function of x

Set $x = p(1-p)$. We have obtained

$$\boxed{E[N] = 3(2x^2 + x + 1)}$$

3.4. Find the value of $p \in (0,1)$ such that $p \mapsto \mathbf{E}[N]$ is maximal.

(i) We have found

$$E[N] = f(x)$$

with

$$f(x) = 3(2x^2 + x + 1)$$

Since $x \mapsto f(x)$ is an increasing function, $f(x)$ is maximal if x is maximal

(ii) We have $x = p(1-p)$. Thus

$p \mapsto x(p)$ maximal when $p = \frac{1}{2}$

Therefore

$$\boxed{p \mapsto E[N] \text{ maximal when } p = \frac{1}{2}}$$