MA/STAT 416 Fall 2024
Probability Theory

Midterm

- You can use a calculator.
- A 2 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 60 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

Name:

Purdue ID:

Problem 1. We throw a pair of dice until the sum of faces shows either 5 or 7.

1.1. Specify the state space S related to this experiment.

Solution:

1.2. Let S_j be the random variable defined by $S_j =$ "Sum of the 2 faces for the j-th roll". Find the pmf of S_j.

Solution:

1.3. We call A the event "5 occurs before 7". We also call A_n the event "5 occurs before 7, and this happens exactly on the n-th roll". Express A_n in terms of the random variables S_j for $j \leq n$.

Solution:

1.4. Compute $\mathbf{P}(A)$ by using the relation $A = \cup_{n=1}^{\infty} A_n$. Justify your answers thanks to the previous questions.

Solution:

1.5. We now define 3 events:

$$F_5 = \{S_1 = 5\}, \qquad F_7 = \{S_1 = 7\}, \qquad F_{\neq} = \{S_1 \notin \{5,7\}\}$$

Compute $\mathbf{P}(A)$ by conditioning on F_5, F_7 and $F_{\neq}$.

Solution:

Problem 2. In order to detect whether a suspect is lying, the police sometimes uses polygraphs. Let $A = \{\text{polygraph indicates lying}\}$ and $B = \{\text{the suspect is lying}\}$. If the suspect is lying, there is a 88% chance of detecting it; if the suspect is telling the truth, 86% of time the polygraph will confirm it. We assume that 1% of the time the suspects lie.

2.1. If the result of the polygraph shows that the suspect is lying, what is the chance that this person is really lying?

Solution:

Problem 3. Two teams play a series of games that ends when one of them has won 3 games. Suppose that each game played is, independently, won by team A with probability $p \in (0, 1)$. We call N the number of games played.

3.1. What is the state space for the random variable N?

Solution:

3.2. Compute the pmf for the random variable N.

Solution:

3.3. Compute $\mathbf{E}[N]$. *Note:* One can express the result as a function of $x = p(1-p)$.

Solution:

3.4. Find the value of $p \in (0,1)$ such that $p \mapsto \mathbf{E}[N]$ is maximal.

Solution: