$\boxed{\text{A problem from Final Spring 20}}$

$1/2$

Problem 5. Consider two independent random variables (X_1, X_2) such that X_1 follows an exponential distribution with mean 2, and X_2 follows a uniform distribution on $[0, 2\pi]$. Let $Y_1 = \sqrt{X_1}\cos(X_2)$ and $Y_2 = \sqrt{X_1}\sin(X_2)$. Show that Y_1 and Y_2 are independent standard normal variables.

Density of $X = (X_1, X_2)$

$$f_X(x_1, x_2)$$

$$= \frac{1}{2} e^{-\frac{x_1}{2}} \mathbb{1}_{(0,\infty)}(x_1) \times \frac{1}{2\pi} \mathbb{1}_{(0,2\pi)}(x_2)$$

$$= \frac{1}{4\pi} \exp\left(\frac{x_1}{2}\right) \mathbb{1}_{(0,\infty)}(x_1) \, \mathbb{1}_{(0,2\pi)}(x_2)$$

Function g we set

$$g(x_1, x_2) = \Big(\underbrace{\sqrt{x_1}\cos(x_2)}_{y_1}, \underbrace{\sqrt{x_1}\sin(x_2)}_{y_2}\Big)$$

Inverting g we have

$$x_1 = y_1^2 + y_2^2$$

$$x_2 = \tan^{-1}(y_2/y_1)$$

$$(y_1, y_2) \in \mathbb{R}^2$$

Computing $f_X(g^{-1}(y))$ Recall

$$f_X(x) = \frac{1}{4\pi} \exp\left(\frac{x_1}{2}\right) \mathbb{1}_{(0,\infty)}(x_1)\, \mathbb{1}_{(0,2\pi)}(x_2)$$

Thus

$$f_X(g^{-1}(y)) = \frac{1}{4\pi} \exp\left(-\frac{(y_1^2+y_2^2)}{2}\right)$$

Computing $J(y)$ we have

$$x_1 = y_1^2 + y_2^2$$

$$x_2 = \tan^{-1}(y_2/y_1)$$

Thus

$$J(y) = \left| \det \begin{pmatrix} 2y_1 & \frac{-y_2}{y_1^2+y_2^2} \\ 2y_2 & \frac{y_1}{y_1^2+y_2^2} \end{pmatrix} \right|$$

$$= 2$$

Conclusion

$$f_Y(y) = f_X(g^{-1}(y))\, J(y)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}(y_1^2+y_2^2)\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_1^2}{2}\right) \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y_1^2}{2}\right)$$