MA 416 - PROBABILITY

REVIEW PROBLEMS - FINAL

Problem 1. At Dr Gesund's office, the waiting time T is modeled by an exponential random variable with mean 10mn. Today the office proposes the following deal: if your waiting time is less than 20mn, you pay the full amount of your visit. Otherwise, you get reimbursed your waiting time minus 20. We call X the amount which is reimbursed by the office. Find the cdf of X. Then find the probability that you get reimbursed twice in 5 visits.

① $T \sim \mathcal{E}(\lambda)$, $\lambda = \frac{1}{10} = \frac{1}{E[T]}$ $\quad f_T(x) = \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)$

$X = (T-20)\, \mathbb{1}_{(T>20)}$

Thus (i) If $x<0$, then $F_X(x) = P(X \le x) = 0$

(ii) $F_X(0) = P(X=0) = P(T \le 20)$

$= \int_0^{20} \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)\, dx$

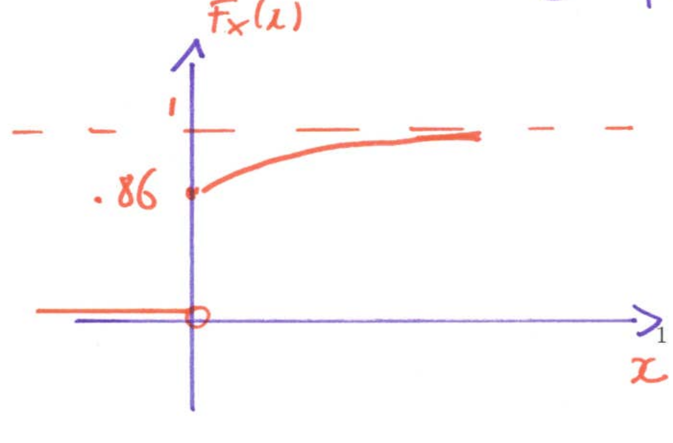
$= \int_0^{20} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x} \Big|_0^{20}$

$= 1 - e^{-\lambda \cdot 20} = 1 - e^{-2} = .86$

(iii) If $x>0$, $F_X(x) = \mathbb{P}(X \le x)$

$= \mathbb{P}(T-20 \le x) = \mathbb{P}(T \le 20+x)$

$= 1 - e^{-\lambda(20+x)}$



(2) For each visit i, $i = 1, \dots, 5$, we set

$$Y_i = \mathbb{1}(\text{reimbursed at visit } i)$$

Then Y_i are i.i.d with dist $\mathcal{B}(p)$

with $p = \mathbb{P}(X > 0) = \mathbb{P}(T > 20)$

$$= e^{-\lambda \times 20} = e^{-2} \simeq .14$$

Set

$$Z = \sum_{i=1}^{5} Y_i = \# \text{ times we get reimbursed in 5 visits}$$

Then $Z \sim \text{Bin}(5, p)$

We wish to compute

$$\mathbb{P}(Z = 2) = \binom{5}{2} p^2 (1-p)^3$$

$$\simeq .12$$

Problem 2. Let X_1, X_2 be two independent variables with common distribution $\mathcal{E}(\lambda)$. Find the density of $\frac{X_1}{X_1+X_2}$.

Let $\varphi \in C_b(\mathbb{R})$. Then

① $E[\varphi(Z)] = E[\varphi(\frac{X_1}{X_1+X_2})]$

Density for (X_1, X_2): $f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

$= \lambda e^{-\lambda x_1} \mathbb{1}_{\mathbb{R}_+}(x_1)$

$\times \lambda e^{-\lambda x_2} \mathbb{1}_{\mathbb{R}_+}(x_2)$

Thus $f(x_1, x_2) = \lambda^2 e^{-\lambda(x_1+x_2)} \mathbb{1}_{\mathbb{R}_+^2}(x_1, x_2)$

and

$E[\varphi(Z)] = \int_{\mathbb{R}^2} \varphi\left(\frac{x_1}{x_1+x_2}\right) \lambda^2 e^{-\lambda(x_1+x_2)} \mathbb{1}_{\mathbb{R}_+^2}(x_1,x_2)\, dx_1\, dx_2$

$= \int_0^\infty \int_0^\infty \varphi\left(\frac{x_1}{x_1+x_2}\right) \lambda^2 e^{-\lambda(x_1+x_2)}$

② CV: $z = \frac{x_1}{x_1+x_2}$ $\quad \omega = x_1 + x_2$. Then

$x_1 = z\omega \qquad x_2 = \omega - x_1 = \omega - z\omega = (1-z)\omega$

$J = \left| \begin{vmatrix} \frac{\partial x_1}{\partial \omega} & \frac{\partial x_2}{\partial \omega} \\ \frac{\partial x_1}{\partial z} & \frac{\partial x_2}{\partial z} \end{vmatrix} \right| = \left| \begin{vmatrix} z & 1-z \\ \omega & -\omega \end{vmatrix} \right| = |-\omega|$

$= \omega$

absolute value

Bounds $0 < \omega < \infty \qquad 0 < z < 1$

② cv-ctd

We get

$$E[\varphi(Z)] = \lambda^2 \int_0^1 dz\, \varphi(z) \int_0^\infty w e^{-\lambda w}\, dw$$

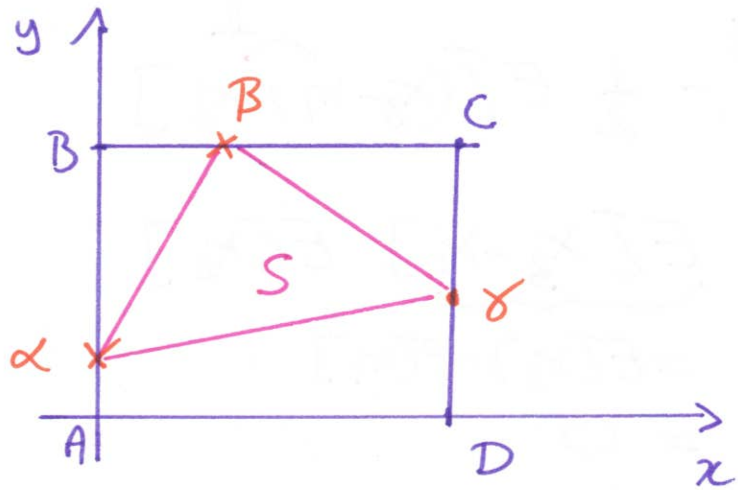
$$= \lambda \int_0^1 dz\, \varphi(z) \underbrace{\int_0^\infty w \lambda e^{-\lambda w}\, dw}_{\substack{E[Y] \text{ with } Y \sim \mathcal{E}(\lambda) \\ = \frac{1}{\lambda}}}$$

$$= \int_0^1 \varphi(z)\, dz$$

$$= \int_{\mathbb{R}} \varphi(z)\, \underbrace{\mathbb{1}_{[0,1]}(z)}_{\text{density of } Z}\, dz$$

Conclusion: $Z \sim \mathcal{U}([0,1])$

Problem 3. Let $ABCD$ be a square with the area 1. Let α, β, γ be random points on $\overline{AB}, \overline{BC}, \overline{CD}$, respectively. Let S be the area of the triangle $\alpha\beta\gamma$. Find $\mathbf{E}[S]$.



Use the formula

$$S = \frac{1}{2} \left| \overrightarrow{\alpha\gamma} \times \overrightarrow{\alpha\beta} \right|$$

Moreover $\alpha = (0, X_1)$ $\beta = (X_2, 1)$

$\gamma = (1, X_3)$

with X_1, X_2, X_3 are $\perp\!\!\!\perp$ $\mathcal{U}([0,1])$

Then

$$\left| \overrightarrow{\alpha\gamma} \times \overrightarrow{\alpha\beta} \right| = \begin{vmatrix} 1 & X_2 \\ X_3 - X_1 & 1 - X_1 \end{vmatrix}$$

$$= 1 - X_1 - (X_3 - X_1) X_2$$

and

$$E[S] = \frac{1}{2} E[1 - X_1 - (X_3 - X_1) X_2]$$

$$= \frac{1}{2} \{ 1 - E[X_1] - E[(X_3 - X_1) X_2] \}$$

We get

$$E[S] = \frac{1}{2} E[1 - x_1 - (x_3 - x_1) x_2]$$

$$= \frac{1}{2} - \frac{1}{2} \overset{\frac{1}{2}}{\overbrace{E[x_1]}} - \frac{1}{2} E[(x_3 - x_1) \overset{\perp}{\frown} x_2]$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{2} \underbrace{E[x_3 - x_1]}_{\substack{= E[x_3] - E[x_1] \\ = 0}} E[x_2]$$

$$= \frac{1}{4}$$

Problem 4. Let U_1, U_2 be two independent variables with common distribution $\mathcal{U}([0,1])$. Their Box-Muller transform can be written as

$$X_1 = (-2\ln(U_1))^{1/2}\cos(2\pi U_2), \qquad X_2 = (-2\ln(U_1))^{1/2}\sin(2\pi U_2).$$

Prove that X_1, X_2 are two independent variables with common distribution $\mathcal{N}(0,1)$.

1 Let $\varphi \in C_b(\mathbb{R}^2)$. Then

$$E[\varphi(X_1, X_2)] = E[\varphi((-2\ln(U_1))^{\frac{1}{2}}\cos(2\pi U_2), (-2\ln(U_1))^{\frac{1}{2}}\sin(2\pi U_2))]$$

Density of $(U_1, U_2) = \mathbb{1}_{[0,1]^2}(u_1, u_2)$. Thus

$$E[\varphi(X_1, X_2)]$$
$$= \int_{[0,1]^2} \varphi((-2\ln(u_1))^{\frac{1}{2}}\cos(2\pi u_1), (-2\ln u_1)^{\frac{1}{2}}\sin(2\pi u_2))\, du_1\, du_2$$

2 CV: $x = (-2\ln(u_1))^{\frac{1}{2}}\cos(2\pi u_2)$
$y = (-2\ln(u_1))^{\frac{1}{2}}\sin(2\pi u_2)$

Then $u_1 = e^{-\frac{1}{2}(x^2+y^2)}$ $\quad u_2 = \frac{1}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$

and

$$J = \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}$$

Bounds: $x, y \in \mathbb{R}$

We get

$E[\varphi(X_1, X_2)]$

$$= \int_{\mathbb{R}^2} \varphi(x,y) \underbrace{\frac{e^{-\frac{1}{2}(x^2+y^2)}}{2\pi}}_{f(x,y),\ \text{density of } (X_1, X_2)} dx\, dy$$

Rmk.

$$f(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}_{\substack{\text{density of} \\ \mathcal{N}(0,1)}} \times \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}}_{\substack{\text{density of} \\ \mathcal{N}(0,1)}}$$

Conclusion:

X_1, X_2 are $\perp\!\!\!\perp$ and $\mathcal{N}(0,1)$

Problem 5. The number of patients arriving at a hospital from 2pm to 3pm with severe symptoms follows a Poisson distribution with mean 1. The hospital ressources are enough to take care of 3 of these patients maximum. What is the probability that the hospital ressources are reached on a given day from 2pm to 3pm? What is the probability that the hospital ressources are reached more than twice on a given week from 2pm to 3pm?

Let X = # severe patients coming from 2pm to 3p.

Then $X \sim \mathcal{P}(\lambda)$, $\lambda = 1$

1 We wish to compute

$$\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X \leq 2)$$

$$= 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right) = 1 - \frac{5}{2}e^{-1}$$

$$\simeq .08 \equiv p$$

2 Y = # times ressources are reached in 1 week

Then Y = # successes in Bernoulli trial

and $Y \sim Bin(7, p)$

We wish to compute

$$\mathbb{P}(Y > 2) = 1 - \mathbb{P}(Y \leq 2)$$

$$= 1 - \mathbb{P}(Y=0) - \mathbb{P}(Y=1) - \mathbb{P}(Y=2)$$

$$= 1 - \left[(1-p)^7 + \binom{7}{1} p (1-p)^6 + \binom{7}{2} p^2 (1-p)^5\right]$$

$$= .014$$