Review - Midterm 24

An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that with probability $\frac{n}{n+m}$ they are all white.

(i) <u>Simple sample space</u>

We can take

$$S = \{W, B\}^{n+m}$$

Note : here the probability is not uniform

(ii) <u>Some events</u> we define

WW = " Only white left at some point "

WL = " Last ball picked is white "

Then

$$WL = WW \; (!) \implies \boxed{\mathbb{P}(WL) = \mathbb{P}(WW)}$$

(iii) Heuristic computation of $\mathbb{P}(WL)$

Last ball is uniformly picked among n+m balls

$\Rightarrow$ $\mathbb{P}(WL) = \mathbb{P}(\text{last ball is } W)$

$$\boxed{\mathbb{P}(WL) = \frac{n}{n+m}}$$

(iv) With conditioning

Define

$W(n,m)$ = Last ball white with initially n white and m black

W_1 = first ball picked is white

B_1 = first ball picked is black

Then

$$\mathbb{P}(W(n,m)) = \mathbb{P}(W(n,m) \mid W_1)\,\mathbb{P}(W_1) + \mathbb{P}(W(n,m) \mid B_1)\,\mathbb{P}(B_1)$$

$$= \mathbb{P}(W(n-1,m))\,\frac{n}{n+m} + \mathbb{P}(W(n,m-1))\,\frac{m}{n+m}$$

set

$$f(n,m) = \mathbb{P}(W(n,m))$$

we have proved

$$\begin{cases} (n+m)\, f(n,m) = n\, f(n-1,m) + m\, f(n,m-1) \\ f(0,1) = 0 \quad , \quad f(1,0) = 1 \end{cases} \quad (1)$$

The solution to the system (1) is

$$\boxed{f(n,m) = \frac{n}{n+m}}$$

Each night different meteorologists give us the probability that it will rain the next day. To judge how well these people predict, we will score each of them as follows: If a meteorologist says that it will rain with probability p, then he or she will receive a score of

$$\begin{cases} 1-(1-p)^2, & \text{if it rains} \\ 1-p^2, & \text{if it does not rain} \end{cases}$$

We will then keep track of scores over a certain time span and conclude that the meteorologist with the highest average score is the best predictor of weather. Suppose now that a given meteorologist is aware of our scoring mechanism and wants to maximize his or her expected score. If this person truly believes that it will rain tomorrow with probability p^*, what value of p should he or she assert so as to maximize the expected score?

(i) Definition of a r.v set

X = score obtained by meteorologist

Then

$$X \in \{ 1-(1-p)^2 , 1-p^2 \}$$

This r.v has pmf

$$\mathbb{P}(X = 1-(1-p)^2) = p^*$$

$$\mathbb{P}(X = 1-p^2) = 1-p^*$$

(ii) <u>Expected value</u> we have

$$\mathbb{E}[X] = \sum_{k \in \{1-(1-p)^2,\, 1-p^2\}} k\, p(k)$$

$$= (1-(1-p)^2)\, p^* + (1-p^2)(1-p^*)$$

$$\equiv \varphi(p)$$

(iii) <u>Optimization</u> we compute

$$\varphi'(p) = 2(1-p)\,p^* - 2p\,(1-p^*)$$

$$= 2\{\, p(-p^* - (1-p^*)) + p^* \}$$

$$= 2\,(p^* - p)$$

Hence

$$\varphi'(p) = 0 \iff p = p^*$$

we get a maximum for

$$\boxed{p = p^*}$$