MA 416 - PROBABILITY

REVIEW PROBLEMS - MIDTERM

Problem 1. [Ross 9th, Chapter 1, Problem 31]. If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?

(1) Start with 2^{nd} question: we wish to find x_1, x_2, x_3, x_4 s.t. x_i integer, $x_i \geq 1$ and $\sum_{i=1}^{4} x_i = 8$. This is a stars & bars problem: place 3 bars between the 8 stars below

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3 bars in the 7 intervals : $\binom{7}{3} = 35$ possibilities

(2) First question: same problem with $x_i \geq 0$. We map this to a stars & bars problem by setting $y_i = x_i + 1$. We get $\sum_{i=1}^{4} y_i = 12$ and $y_i \geq 1$.

$$\binom{11}{3} = 165 \text{ possibilities}$$

Problem 2. [Ross 9th, Chapter 2, Problem 15]. In a poker game, what is the probability of being dealt two pairs?

Probability: on $S = \{\text{hands of 5 cards}\}$, we consider the uniform probability. We have $|S| = \binom{52}{5}$.

Two pairs: of the form a a b b c
Set $A =$ (two pairs dealt). We have

$$|A| = \binom{13}{2} \times \binom{4}{2} \times \binom{44}{1}$$

pick (a, b) among 13 cards

pick 2 colors among 4

pick 1 card among remaining 44 cards

Thus $P(A) = \dfrac{|A|}{|S|} = .048$

Problem 3. We have two classes of people: those who are accident prone and those who are not. Accident prone people have a probability .4 of accident in a one-year period. Those who are not accident prone have a probability .2 of accident in a one-year period. 30% of the population is accident prone. What is the probability that a new policyholder will have an accident within her/his second year of purchasing a policy if we know she/he had an accident in his first year?

Set $A_1 =$ accident in 1^{st} year
$A_2 -$ accident in 2^{nd} year
$A =$ accident prone

$P(A_1|A) = .4$
$P(A_1|A^c) = .2$
$P(A) = .3$
$P(A^c) = .7$

We wish to compute $P(A_2|A_1)$. We have

$$P(A_2|A_1) = \frac{P(A_1A_2)}{P(A_1)}$$

Then

$$P(A_1) = P(A_1|A)P(A) + P(A_1|A^c)P(A^c) =$$
$$= .4 \times .3 + .2 \times .7 = .26$$

We assume conditional independence. Then

$$P(A_1A_2) = P(A_1A_2|A)P(A) + P(A_1A_2|A^c)P(A^c)$$
$$\overset{\text{cdt}\perp}{=} P(A_1|A)P(A_2|A)P(A) + P(A_1|A^c)P(A_2|A^c)P(A^c)$$
$$= (.4)^2 \cdot .3 + (.2)^2 \cdot .7 = .076$$

and

$$P(A_2|A_1) = \frac{.076}{.26} = .29$$

Problem 4. We draw 5 t-shirts in a very large lot. There are 3 sizes of t-shirts (say 1,2 and 3), each one with equal probability. We call S_i the event that we get at least one t-shirt of size i. Find $\mathbf{P}(A_1 \cup A_2)$. Compute $\mathbf{P}(A_1 A_2)$.

① Compute $P(A_1)$

$P(A_1) = 1 - P(A_1^c) = 1 - (1-\frac{1}{3})^5 = 0.868$

In the same way, $P(A_2) = 0.868$

② Compute $P(A_1 \cup A_2)$

$P(A_1 \cup A_2) = 1 - P((A_1 \cup A_2)^c) = 1 - P(A_1^c A_2^c)$

$= 1 - (1-\frac{2}{3})^5 = 0.996$

③ $P(A_1 A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$

$= 2 \times 0.868 - 0.996$

$= 0.74$

Problem 5. [Ross 9th, Chapter 3, Problem 74]. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.

Sample space: $S = \{x_i \,;\, i \geq 1\}$ with $x_i \in \{2, \dots, 12\}$

Probability: For each $n \geq 1$ and $a_1, \dots, a_n$

$$P(\{x_i \,;\, x_1 = a_1, \dots, x_n = a_n\}) = \prod_{i=1}^{n} g(a_i)$$

with $g(2) = \frac{1}{36}$, $g(3) = \frac{2}{36}$, $\dots$, $g(12) = \frac{1}{36}$

Set $\bar{p}_6 = P(\text{generic throw} \neq 6) = \frac{30}{36} = \frac{15}{18}$

$\bar{p}_9 = P(\text{generic throw} \neq 9) = \frac{32}{36} = \frac{8}{9}$

$p_9 = P(\text{generic throw} = 9) = \frac{1}{9}$

F_A = "Final throw by player A"

Then

$$P(F_A) = \sum_{k=0}^{\infty} P(F_A \cap (\text{last throw is } \# 2k+1))$$

$$= \sum_{k=0}^{\infty} P(x_1 \neq 9, x_2 \neq 6, \dots, x_{2k-1} \neq 9, x_{2k} \neq 6, x_{2k+1} = 9)$$

$$= \sum_{k=0}^{\infty} (\bar{p}_9)^k (\bar{p}_6)^k \, p_9$$

$$= \frac{p_9}{1 - \bar{p}_6 \, \bar{p}_9}$$

$$= \frac{3}{7}$$