Limit theorems

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Mostly taken from *A first course in probability*
by S. Ross

Outline

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1. Weak law of large numbers

2. Central limit theorem

Weak law of large numbers

Theorem 1.

Consider

- A sequence $\{X_i;\ i \geq 1\}$ of i.i.d random variables
- Write $\mathbf{E}[X_i] = \mu$
- Set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mathbf{P}\left(|\bar{X}_n - \mu| > \varepsilon \right) = 0$$

Outline

DeMoivre-Laplace theorem (repeated)

Theorem 2.

Let

- $n \geq 1, \ p \in (0, 1)$
- $X_n \sim \text{Bin}(n, p)$
- $a < b$

Then

$$\lim_{n \to \infty} \mathbf{P} \left(a < \frac{X_n - np}{(np(1-p))^{1/2}} < b \right) = \Phi(b) - \Phi(a)$$

Another way to write De Moivre's theorem

Theorem 3.

Consider

- A sequence $\{Y_i;\ i \geq 1\}$ of indep. $\mathcal{B}(p)$ random variables
- We have $\mathbf{E}[Y_i] = p$ and $\mathbf{Var}(Y_i) = p(1-p) \equiv \sigma^2$
- Set

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbf{P}\left(a < \sqrt{n}\left(\frac{\bar{Y}_n - p}{\sigma}\right) < b\right) = \Phi(b) - \Phi(a)$$

Central limit theorem

Theorem 4.

Consider

- A sequence $\{X_i;\ i \geq 1\}$ of i.i.d random variables
- Write $\mathbf{E}[X_i] = \mu$ and $\mathbf{Var}(X_i) = \sigma^2$
- Set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Then for any $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \mathbf{P}\left(a < \sqrt{n} \left(\frac{\bar{X}_n - \mu}{\sigma} \right) < b \right) = \Phi(b) - \Phi(a)$$

Problem 8.5 (1)

Situation: We have

- Fifty numbers rounded off to the nearest integer and then summed
- The individual round-off errors are uniformly distributed over $(-0.5, 0.5)$

Question:

Approximate the probability that the resultant sum differs from the exact sum by more than 3.

Problem 8.5 (2)

Model: Set

$$X_i \equiv i\text{-th error}$$

We wish to find

$$\mathbf{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right)$$

Law of X_i: We have

- X_i's i.i.d
- $X_i \sim \mathcal{U}([-0.5, 0.5])$
- $\mathbf{E}[X_i] = 0$, and $\mathbf{Var}(X_i) = \sigma^2 = \frac{1}{12}$

Problem 8.5 (3)

Application of CLT: Write

$$\begin{aligned}
\mathbf{P}\left(\left|\sum_{i=1}^{n} X_i\right| > 3\right) &= \mathbf{P}\left(\left|\sqrt{n}\,\bar{X}_n\right| > \frac{3}{\sqrt{50}}\right) \\
&= \mathbf{P}\left(\left|\sqrt{n}\,\frac{\bar{X}_n}{\sigma}\right| > \frac{3\sqrt{12}}{\sqrt{50}}\right) \\
&\overset{CLT}{\simeq} \mathbf{P}\left(|Z| > 1.47\right) \\
&= .14
\end{aligned}$$

Problem 8.14 (1)

Situation: We consider

- A certain component, which is critical to the operation of an electrical system and must be replaced immediately upon failure
- For the critical component

$$\text{Mean} = 100 \text{ h, and Standard deviation} = 30\text{h}$$

Question:
How many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least .95?

Problem 8.14 (2)

Model: Set

$$X_i \equiv \text{lifespan of } i\text{-th component}$$

We wish to find n minimal such that

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i > 2000\right) > .95$$

Law of X_i: We have

- X_i's i.i.d
- $\mathbf{E}[X_i] \equiv \mu = 100$, and $\sqrt{\mathbf{Var}(X_i)} \equiv \sigma = 30$

Problem 8.14 (3)

Application of CLT: Write

$$
\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{n} X_i > 2000\right) &= \mathbf{P}\left(\bar{X}_n > \frac{2000}{n}\right) \\
&= \mathbf{P}\left(\sqrt{n}\,\frac{\left(\bar{X}_n - \mu\right)}{\sigma} > \sqrt{n}\,\frac{\left(\frac{2000}{n} - 100\right)}{30}\right) \\
&\overset{CLT}{\simeq} \mathbf{P}\left(Z > \sqrt{n}\,\frac{\left(\frac{200}{n} - 10\right)}{3}\right)
\end{aligned}
$$

Problem 8.14 (4)

Reading the Gaussian table: Since

$$\mathbf{P}(Z > -1.64) \simeq .95,$$

we are looking for a n minimal such that

$$\sqrt{n}\,\frac{\left(\frac{200}{n} - 10\right)}{3} \leq -1.64 \tag{1}$$

Corresponding 2nd order equation: Setting $\sqrt{n} = \frac{1}{x}$, we get

$$200x^2 + 4.92x - 10 \leq 0$$

This yields

$$n \geq 23$$