Random variables

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Probability - MA 416

Mostly taken from *A first course in probability*
by S. Ross

Outline

Outline

1. Random variables
2. Discrete random variables
3. Expected value
4. Expectation of a function of a random variable
5. Variance
6. The Bernoulli and binomial random variables
7. The Poisson random variable
8. Other discrete random variables
9. Expected value of sums of random variables
10. Properties of the cumulative distribution function

Introduction

Experiment: tossing 3 coins

Model:

$$S = \{h, t\}^3,\ \mathbf{P}(\{s\}) = \frac{1}{8} \text{ for all } s \in S$$

Result of the experiment: we are interested in the quantity

$$X(s) = \text{"\# Heads obtained when } s \text{ is realized"}$$

Introduction (2)

Table for the outcomes:

s	$X(s)$	s	$X(s)$
(t, t, t)	0	(h, t, t)	1
(t, t, h)	1	(h, t, h)	2
(t, h, t)	1	(h, h, t)	2
(t, h, h)	2	(h, h, h)	3

Introduction (3)

Information about X:

X is considered as an application, i.e.

$$X : S \to \{0, 1, 2, 3\}.$$

Then we wish to understand sets like

$$X^{-1}(\{2\}) = \{(t, h, h), (h, t, h), (h, h, t)\}$$

or quantities like

$$\mathbf{P}\left(X^{-1}(\{2\})\right) = \frac{3}{8}.$$

This will be formalized in this chapter

Example: time of first success (1)

Experiment:

- Coin having probability p of coming up heads
- Independent trials: flipping the coin
- Stopping rule: either H occurs or n flips made

Random variable:

$$X = \# \text{ of times the coin is flipped}$$

State space:

$$X \in \{1, \ldots, n\}$$

Example: time of first success (2)

Probabilities for $j < n$:

$$\mathbf{P}(X = j) = \mathbf{P}\left(\{(t, \ldots, t, h)\}\right) = (1-p)^{j-1} p$$

Probability for $j = n$:

$$\mathbf{P}(X = n) = \mathbf{P}\left(\{(t, \ldots, t, h); (t, \ldots, t, t)\}\right) = (1-p)^{n-1}$$

Example: time of first success (3)

Checking the sum of probabilities:

$$\begin{aligned}
\mathbf{P}\left(\bigcup_{j=1}^{n}\{X=j\}\right) &= \sum_{j=1}^{n}\mathbf{P}\left(\{X=j\}\right) \\
&= p\sum_{j=1}^{n-1}(1-p)^{j-1}+(1-p)^{n} \\
&= 1
\end{aligned}$$

Cumulative distribution function

Definition 1.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$

For $x \in \mathbb{R}$ we define

$$F(x) = \mathbf{P}(X \le x)$$

Then the function F is called cumulative distribution function or distribution function

Outline

General definition

Definition 2.

Let

- **P** a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable

Hypothesis: $\mathcal{E}$ is countable, i.e

$$\mathcal{E} = \{x_i; i \geq 1\}$$

Then we say that X is a discrete random variable

Probability mass function

Definition 3.

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i; i \geq 1\}$ countable state space
- $X : S \to \mathcal{E}$ discrete random variable

For $i \geq 1$ we set

$$p(x_i) = \mathbf{P}(X = x_i)$$

Then the probability mass function of X is the family

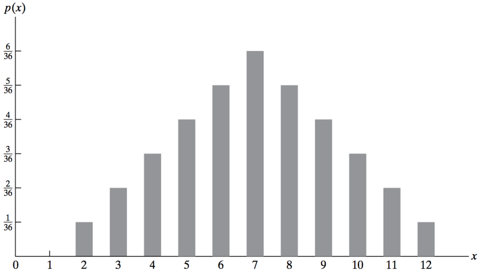
$$\{p(x_i);\ i \geq 1\}$$

Remarks

Sum of the pmf: If p is the pmf of X, then

$$\sum_{i \geq 1} p(x_i) = 1$$

Graph of a pmf: Bar graphs are often used.
Below an example for X = sum of two dice



Example of pmf computation (1)

Definition of the pmf: Let X be a r.v with pmf given by

$$p(i) = c\,\frac{\lambda^i}{i!}, \qquad i \geq 0,$$

where $c > 0$ is a normalizing constant

Question: Compute

1. $\mathbf{P}(X = 0)$
2. $\mathbf{P}(X > 2)$

Example of pmf computation (2)

Computing c: We must have

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Thus

$$c = e^{-\lambda}$$

Computing $\mathbf{P}(X = 0)$: We have

$$\mathbf{P}(X = 0) = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}$$

Example of pmf computation (3)

Computing $\mathbf{P}(X > 2)$: We have

$$\mathbf{P}(X > 2) = 1 - \mathbf{P}(X \leq 2)$$

Thus

$$\mathbf{P}(X > 2) = 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

Cdf for discrete random variables

Proposition 4.

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i;\, i \geq 1\}$ countable state space, with $\mathcal{E} \subset \mathbb{R}$
- $X : S \to \mathcal{E}$ discrete random variable
- F cdf of X and p pmf of X

Then

1. F can be expressed as
$$F(a) = \sum_{i \geq 1;\, x_i \leq a} p(x_i)$$
2. F is a step function

Example of discrete cdf (1)

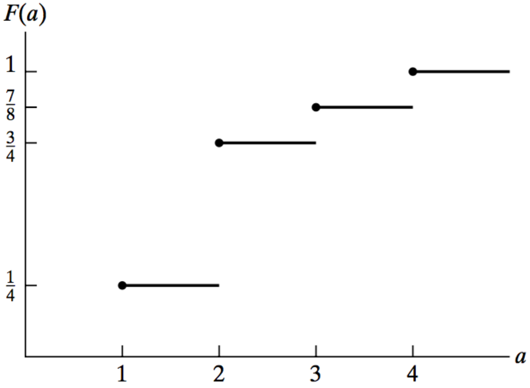
Definition of the random variable:

Consider $X : S \to \{1, 2, 3, 4\}$ given by

$$p(1) = \frac{1}{4}, \quad p(2) = \frac{1}{2}, \quad p(3) = \frac{1}{8}, \quad p(4) = \frac{1}{8}$$

Example of discrete cdf (2)

Graph of F:



Outline

Expected value for discrete random variables

Definition 5.

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i; i \geq 1\}$ countable state space, with $\mathcal{E} \subset \mathbb{R}$
- $X : S \to \mathcal{E}$ discrete random variable
- p pmf of X

Then we define

$$\mathbf{E}[X] = \sum_{i \geq 1} x_i \, \mathbf{P}(X = x_i) = \sum_{i \geq 1} x_i \, p(x_i)$$

Justification of the definition

Experiment:

- Run independent copies of the random variable X
- For i-th copy, the measurement is z_i

Result (to be proved much later):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} z_i = \mathbf{E}[X]$$

Example: dice rolling (1)

Definition of the random variable: we consider

$$X = \text{outcome when we roll a fair dice}$$

Example: dice rolling (2)

Recall: we consider

$$X = \text{outcome when we roll a fair dice}$$

Pmf: We have $\mathcal{E} = \{1, \ldots, 6\}$ and

$$p(1) = \cdots = p(6) = \frac{1}{6}$$

Expected value: We get

$$\mathbf{E}[X] = \sum_{i=1}^{6} i\, p(i) = \frac{1}{6} \sum_{i=1}^{6} i = \frac{7}{2}$$

Example: indicator of an event (1)

Definition of the random variable:

Let A event with $\mathbf{P}(A) = p$ and set

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

Example: indicator of an event (2)

Recall:

Let A event with $\mathbf{P}(A) = p$ and set

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

Pmf:

$$p(0) = 1 - p, \qquad p(1) = p$$

Expected value:

$$\mathbf{E}[\mathbf{1}_A] = p$$

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First attempt of a definition

Problem: Let

- X discrete random variable
- $Y = g(X)$ for a function g

How can we compute $\mathbf{E}[g(X)]$?

First strategy:

- $Y = g(X)$ is a discrete random variable
- Determine the pmf p_Y of Y
- Compute $\mathbf{E}[Y]$ according to Definition 5

First attempt: example (1)

Definition of a random variable X:

Let $X : S \to \{-1, 0, 1\}$ with

$$\mathbf{P}(X = -1) = .2, \quad \mathbf{P}(X = 0) = .5, \quad \mathbf{P}(X = 1) = .3$$

We wish to compute $\mathbf{E}[X^2]$

First attempt: example (2)

Definition of a random variable Y: Set $Y = X^2$.
Then $Y \in \{0, 1\}$ and

$$\begin{aligned} \mathbf{P}(Y = 0) &= \mathbf{P}(X = 0) = .5 \\ \mathbf{P}(Y = 1) &= \mathbf{P}(X = -1) + \mathbf{P}(X = 1) = .5 \end{aligned}$$

First attempt: example (3)

Recall: For $Y = X^2$ we have

$$\mathbf{P}(Y = 0) = .5, \qquad \mathbf{P}(Y = 1) = .5$$

Expected value:

$$\mathbf{E}\left[X^2\right] = \mathbf{E}[Y] = .5$$

Definition of $\mathbf{E}[g(X)]$

Proposition 6.

Let

- X discrete random variable
- p pmf of X
- g real valued function

Then

$$\mathbf{E}\left[g(X)\right] = \sum_{i \geq 1} g(x_i)\, p(x_i) \tag{1}$$

Proof

Values of Y: We set $Y = g(X)$ and

$$\{y_j;\, j \geq 1\} = \text{ values of } g(x_i) \text{ for } i \geq 1$$

Expression for the rhs of (1): gather according to y_j

$$\begin{aligned}
\sum_{i\geq 1} g(x_i)\, p(x_i) &= \sum_{j\geq 1} \sum_{i;\, g(x_i)=y_j} y_j\, p(x_i) \\
&= \sum_{j\geq 1} y_j \sum_{i;\, g(x_i)=y_j} p(x_i) \\
&= \sum_{j\geq 1} y_j\, \mathbf{P}\left(g(X) = y_j\right) \\
&= \sum_{j\geq 1} y_j\, \mathbf{P}\left(Y = y_j\right) \\
&= \mathbf{E}\left[g(X)\right]
\end{aligned}$$

Previous example reloaded

Definition of a random variable X:

Let $X : S \to \{-1, 0, 1\}$ with

$$\mathbf{P}(X = -1) = .2, \quad \mathbf{P}(X = 0) = .5, \quad \mathbf{P}(X = 1) = .3$$

We wish to compute $\mathbf{E}[X^2]$

Application of (1):

$$\mathbf{E}\left[X^2\right] = \sum_{i=-1,0,1} i^2 p(x_i) = .5$$

Example: seasonal product (1)

Situation:

- Product sold seasonally
- Profit b for each unit sold
- Loss ℓ for each unit left unsold
- Product has to be stocked in advance
  $\hookrightarrow$ s units stocked

Random variable:

- $X = \#$ units of product ordered
- Pmf p for X

Question:
Find optimal s in order to maximize profits

Example: seasonal product (2)

Some random variables: We set

$$\begin{aligned} X &= \# \text{ units ordered, with pmf } p \\ Y_s &= \text{profit when } s \text{ units stocked} \end{aligned}$$

Expression for Y_s:

$$Y_s = (b\,X - (s - X)\,\ell)\,\mathbf{1}_{(X \le s)} + s\,b\,\mathbf{1}_{(X > s)}$$

Expression for $\mathbf{E}[Y_s]$:

$$\mathbf{E}\,[Y_s] \;=\; \sum_{i=0}^{s} (b\,i - (s - i)\,\ell)\,p(i) + \sum_{i=s+1}^{\infty} s\,b\,p(i)$$

Example: seasonal product (3)

Simplification for $\mathbf{E}[Y_s]$: We get

$$\mathbf{E}\left[Y_s\right] = s\,b + (b+\ell)\sum_{i=0}^{s}(i-s)\,p(i)$$

Growth of $s \mapsto \mathbf{E}[Y_s]$: We have

$$\mathbf{E}\left[Y_{s+1}\right] - \mathbf{E}\left[Y_s\right] = b - (b+\ell)\sum_{i=0}^{s}p(i)$$

Example: seasonal product (4)

Growth of $s \mapsto \mathbf{E}[Y_s]$ (Ctd): We obtain

$$\mathbf{E}\left[Y_{s+1}\right] - \mathbf{E}\left[Y_s\right] > 0 \quad \Longleftrightarrow \quad \sum_{i=0}^{s} p(i) < \frac{b}{b+\ell} \qquad (2)$$

Optimization:

- The lhs of (2) is $\nearrow$
- The rhs of (2) is constant
- Thus there exists a s^* such that

$$\mathbf{E}\left[Y_0\right] < \cdots < \mathbf{E}\left[Y_{s^*-1}\right] < \mathbf{E}\left[Y_{s^*}\right] > \mathbf{E}\left[Y_{s^*+1}\right] > \cdots$$

Conclusion: s^* leads to maximal expected profit

Expectation and linear transformations

Proposition 7.

Let

- X discrete random variable
- p pmf of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{E}[aX + b] = a\,\mathbf{E}[X] + b$$

Proof

Application of relation (1):

$$\begin{aligned} \mathbf{E}\left[aX+b\right] &= \sum_{i\geq 1} (a\, x_i + b)\; p(x_i) \\ &= a \sum_{i\geq 1} x_i\, p(x_i) + b \sum_{i\geq 1} p(x_i) \\ &= a\,\mathbf{E}\left[X\right] + b \end{aligned}$$

Outline

Definition of variance

Definition 8.

Let

- X discrete random variable
- p pmf of X
- $\mu = \mathbf{E}[X]$

Then we define $\mathbf{Var}(X)$ by

$$\mathbf{Var}(X) = \mathbf{E}\left[(X-\mu)^2\right]$$

Interpretation

Expected value: For a r.v X, $\mathbf{E}[X]$ represents
the mean value of X.

Variance: For a r.v X, $\mathbf{Var}(X)$ represents
the dispersion of X wrt its mean value.

A greater $\mathbf{Var}(X)$ means

- The system represented by X has a lot of randomness
- This system is unpredictable

Standard deviation: For physical reasons, it is better to introduce

$$\sigma_X := \sqrt{\mathbf{Var}(X)}.$$

Interpretation (2)

Illustration (from descriptive stats): We wish to compare the performances of 2 soccer players on their last 5 games

Griezmann	5	0	0	0	0
Messi	1	1	1	1	1

Recall: for a set of data $\{x_i;\ i \leq n\}$, we have
Empirical mean: $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$
Empirical variance: $s_n^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2$
Standard deviation: $s_n = \sqrt{s_n^2}$

On our data set: $\bar{x}_G = \bar{x}_M = 1$ goal/game
$\hookrightarrow$ Same goal average
However, $s_G = 2$ goals/game while $s_M = 0$ goals/game
$\hookrightarrow$ M more reliable (less random) than G

Alternative expression for the variance

Proposition 9.

Let

- X discrete random variable
- p pmf of X
- $\mu = \mathbf{E}[X]$

Then $\mathbf{Var}(X)$ can be written as

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - \mu^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Example: rolling a dice

Random variable:

- $X =$ outcome when one rolls 1 dice

Variance computation: We find

$$\mathbf{E}[X] = \frac{7}{2}, \qquad \mathbf{E}[X^2] = \frac{91}{6}$$

Therefore

$$\mathbf{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Standard deviation:

$$\sigma_X = \sqrt{\frac{35}{12}} \simeq 1.71$$

Variance and linear transformations

Proposition 10.

Let

- X discrete random variable
- p pmf of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{Var}(aX + b) = a^2 \, \mathbf{Var}(X)$$

Outline

Bernoulli random variable (1)

Notation:

$$X \sim \mathcal{B}(p) \text{ with } p \in (0,1)$$

State space:

$$\{0,1\}$$

Pmf:

$$\mathbf{P}(X=0)=1-p, \quad \mathbf{P}(X=1)=p$$

Expected value and variance:

$$\mathbf{E}[X]=p, \qquad \mathbf{Var}(X)=p(1-p)$$

Bernoulli random variable (2)

Use 1, success in a binary game:

- Example 1: coin tossing
  - $X = 1$ if H, $X = 0$ if T
  - We get $X \sim \mathcal{B}(1/2)$
- Example 2: dice rolling
  - $X = 1$ if outcome $= 3$, $X = 0$ otherwise
  - We get $X \sim \mathcal{B}(1/6)$

Use 2, answer yes/no in a poll

- $X = 1$ if a person feels optimistic about the future
- $X = 0$ otherwise
- We get $X \sim \mathcal{B}(p)$, with unknown p

Jacob Bernoulli

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- First law of large numbers
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Binomial random variable (1)

Notation:

$$X \sim \text{Bin}(n, p), \text{ for } n \geq 1,\ p \in (0, 1)$$

State space:

$$\{0, 1, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n$$

Expected value and variance:

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1-p)$$

Binomial random variable (2)

Use 1, Number of successes in a Bernoulli trial:

- Example: Roll a dice 9 times.
- $X = \#$ of 3 obtained
- We get $X \sim \text{Bin}(9, 1/6)$
- $\mathbf{P}(X = 2) = 0.28$

Use 2: Counting a feature in a repeated trial:

- Example: stock of 1000 pants with 10% defects
- Draw 15 times a pant at random
- $X = \#$ of pants with a defect
- We get $X \sim \text{Bin}(15, 1/10)$

Binomial random variable (3)

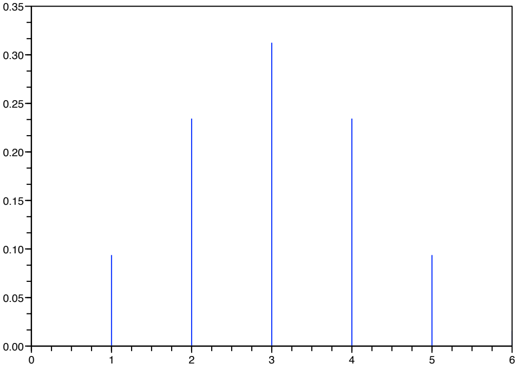


Figure: Pmf for Bin(6; 0.5). x-axis: k. y-axis: $\mathbf{P}(X = k)$

Binomial random variable (4)

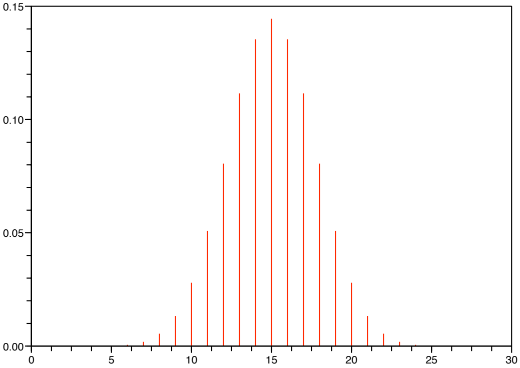


Figure: Pmf for $\text{Bin}(30; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Example: wheel of fortune (1)

Game:

- Player bets on $1, \ldots, 6$ (say 1)
- 3 dice rolled
- If 1 does not appear, loose \$1
- If 1 appear i times, win \$i

Question:
Find average win

Example: wheel of fortune (2)

Binomial random variable:

- Let $X = \#$ times 1 appears
- Then $X \sim \text{Bin}(3, \frac{1}{6})$

Expression for the win: Set $W =$ win. Then

- $W = \varphi(X)$ with
  $\hookrightarrow \varphi(0) = -1$ and $\varphi(i) = i$ for $i = 1, 2, 3$
- Other expression:

$$W = X - \mathbf{1}_{(X=0)}$$

Example: wheel of fortune (3)

Average win:

$$\begin{aligned} \mathbf{E}[W] &= \mathbf{E}[X] - \mathbf{P}(X=0) \\ &= \frac{1}{2} - \left(\frac{5}{6}\right)^3 \\ &= -\frac{17}{216} \end{aligned}$$

Conclusion: The average win is

$$\mathbf{E}[W] \simeq -\$0.079$$

Pmf variations for a binomial r.v

Proposition 11.

Let

- $X \sim \text{Bin}(n, p)$
- $q = \text{Pmf of } X$
- $k^* = \lfloor (n+1)p \rfloor$

Then we have

- $k \mapsto q(k)$ is $\nearrow$ if $k < k^*$
- $k \mapsto q(k)$ is $\searrow$ if $k > k^*$
- Maximum of q attained for $k = k^*$

Proof

Pmf computation: We have

$$\frac{q(k)}{q(k-1)} = \frac{\mathbf{P}(X=k)}{\mathbf{P}(X=k-1)} = \frac{(n-k+1)p}{k(1-p)}$$

Pmf growth: We get

$$\mathbf{P}(X=k) \geq \mathbf{P}(X=k-1) \quad \Longleftrightarrow \quad k \leq (n+1)p$$

Outline

Poisson random variable (1)

Notation:

$$\mathcal{P}(\lambda) \text{ for } \lambda > 0$$

State space:

$$E = \mathbb{N} \cup \{0\}$$

Pmf:

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0$$

Expected value and variance:

$$\mathbf{E}[X] = \lambda, \qquad \mathbf{Var}(X) = \lambda$$

Poisson random variable (2)

Use (examples):

- # customers getting into a shop from 2pm to 5pm
- # buses stopping at a bus stop in a period of 35mn
- # jobs reaching a server from 12am to 6am

Empirical rule:

If $n \to \infty$, $p \to 0$ and $np \to \lambda$, we approximate $\text{Bin}(n, p)$ by $\mathcal{P}(\lambda)$.
This is usually applied for

$$p \leq 0.1 \quad \text{and} \quad np \leq 5$$

Poisson random variable (3)

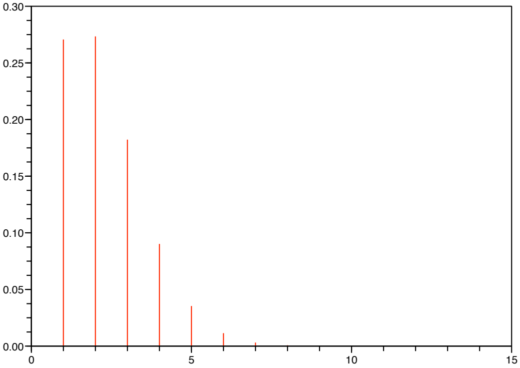


Figure: Pmf of $\mathcal{P}(2)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Poisson random variable (4)

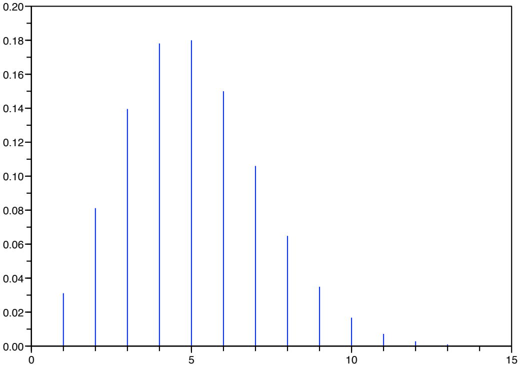


Figure: Pmf of $\mathcal{P}(5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Siméon Poisson

Some facts about Poisson:

- Lifespan: 1781-1840, in $\simeq$ Paris
- Engineer, Physicist and Mathematician
- Breakthroughs in electromagnetism
- Contributions in partial diff. eq
  celestial mechanics, Fourier series
- Marginal contributions in probability

A quote by Poisson:

Life is good for only two things: doing mathematics and teaching it!!

Example: drawing defective items (1)

Experiment:

- Item produced by a certain machine will be defective
  $\hookrightarrow$ with probability .1
- Sample of 10 items drawn

Question:
Probability that the sample contains at most 1 defective item

Example: drawing defective items (2)

Random variable: Let

$$X = \# \text{ of defective items}$$

Then

$$X \sim \text{Bin}(n, p), \quad \text{with} \quad n = 10,\ p = .1$$

Exact probability: We have to compute

$$\begin{aligned} \mathbf{P}(X \leq 1) &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1) \\ &= (0.9)^{10} + 10 \times 0.1 \times (0.9)^9 \\ &= .7361 \end{aligned}$$

Example: drawing defective items (3)

Approximation: We use

$$\text{Bin}(10, .1) \simeq \mathcal{P}(1)$$

Approximate probability: We have to compute

$$\begin{aligned} \mathbf{P}(X \leq 1) &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1) \\ &\simeq e^{-1}(1 + 1) \\ &= .7358 \end{aligned}$$

Poisson paradigm

Situation: Consider

- n events $E_1, \ldots, E_n$
- $p_i = \mathbf{P}(E_i)$
- Weak dependence of the E_i: $\mathbf{P}(E_i E_j) \lesssim \frac{1}{n}$
- $\lim_{n\to\infty} \sum_{i=1}^{n} p_i = \lambda$

Heuristic limit: Under the conditions above we expect that

$$X_n = \sum_{i=1}^{n} \mathbf{1}_{E_i} \to \mathcal{P}(\lambda) \tag{3}$$

Example: matching problem (1)

Situation:

- n men take off their hats
- Hats are mixed up
- Then each man selects his hat at random
- Match: if a man selects his own hat

Question: Compute

- $\mathbf{P}(E_k)$ with $E_k =$ "exactly k matches"

Example: matching problem (2)

Fact: Using heavy combinatorics, one can prove

$$\mathbf{P}(E_k) = \frac{1}{k!} \sum_{j=2}^{n-k} \frac{(-1)^j}{j!}$$

Thus

$$\lim_{n\to\infty} \mathbf{P}(E_k) = \frac{e^{-1}}{k!}$$

New events: We set

$$G_i = \text{"Person } i \text{ selects his own hat"}$$

Example: matching problem (3)

Probabilities for G_i: We have

$$\mathbf{P}(G_i) = \frac{1}{n}, \qquad \mathbf{P}\left(G_i \middle| G_j\right) = \frac{1}{n-1}$$

Random variable of interest:

$$X = \sum_{i=1}^{n} \mathbf{1}_{G_i} \quad \Longrightarrow \quad \mathbf{P}(E_k) = \mathbf{P}\left(X = k\right)$$

Poisson paradigm: From (3) we have $X \simeq \mathcal{P}(1)$. Therefore

$$\mathbf{P}(E_k) = \mathbf{P}\left(X = k\right) \simeq \mathbf{P}\left(\mathcal{P}(1) = k\right) = \frac{e^{-1}}{k!}$$

Outline

Geometric random variable

Notation:

$$X \sim \mathcal{G}(p), \qquad \text{for } p \in (0,1)$$

State space:

$$E = \mathbb{N} = \{1, 2, 3, \dots\}$$

Pmf:

$$\mathbf{P}(X = k) = p\,(1-p)^{k-1}, \quad k \geq 1$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}$$

Geometric random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until first success

Example: dice rolling

- Set $X = $ 1st roll for which outcome $= 6$
- We have $X \sim \mathcal{G}(1/6)$

Computing some probabilities for the example:

$$\mathbf{P}(X = 5) = \left(\frac{5}{6}\right)^4 \frac{1}{6} \simeq 0.08$$

$$\mathbf{P}(X \geq 7) = \left(\frac{5}{6}\right)^6 \simeq 0.33$$

Geometric random variable (3)

Computation of $\mathbf{E}[X]$: Set $q = 1 - p$. Then

$$\begin{aligned} \mathbf{E}[X] &= \sum_{i=1}^{\infty} i q^{i-1} p \\ &= \sum_{i=1}^{\infty} (i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\ &= q\, \mathbf{E}[X] + 1 \end{aligned}$$

Conclusion:

$$\mathbf{E}[X] = \frac{1}{p}$$

Tail of a geometric random variable

Proposition 12.

Let

- $X \sim \mathcal{G}(p)$
- $n \geq 1$

Then we have

$$\mathbf{P}\left(X \geq n\right) = (1-p)^{n-1}$$

Negative binomial random variable (1)

Notation:

$$X \sim \text{Nbin}(r, p), \text{ for } r \in \mathbb{N}^*, \; p \in (0,1)$$

State space:

$$\{r, r+1, r+2 \dots\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r \, (1-p)^{k-r}, \quad k \geq r$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{r}{p}, \qquad \mathbf{Var}(X) = \frac{r(1-p)}{p^2}$$

Negative binomial random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until r successes

Justification:

$$(X = k) = (r-1 \text{ successes in } (k-1) \text{ 1st trials}) \cap (k\text{-th trial is a success})$$

Thus

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Moments of negative binomial random variable

Proposition 13.

Let

- $X \sim \text{Nbin}(r, p)$, for $r \geq 1$, $p \in (0, 1)$
- $Y \sim \text{Nbin}(r + 1, p)$
- $l \geq 1$

Then

$$\mathbf{E}\left[X^l\right] = \frac{r}{p}\,\mathbf{E}\left[(Y - 1)^{l-1}\right]$$

Proof (1)

Definition of the l-th moment: We have

$$\mathbf{E}\left[X^l\right] = \sum_{k=r}^{\infty} k^l \binom{k-1}{r-1} p^r (1-p)^{k-r}$$

Relation for combination numbers:

$$k\binom{k-1}{r-1} = r\binom{k}{r}$$

Consequence:

$$\mathbf{E}\left[X^l\right] = r\sum_{k=r}^{\infty} k^{l-1} \binom{k}{r} p^r (1-p)^{k-r}$$

Proof (2)

Recall:

$$\mathbf{E}\left[X^l\right] = r\sum_{k=r}^{\infty} k^{l-1}\binom{k}{r} p^r (1-p)^{k-r}$$

From r to $r+1$:

$$\mathbf{E}\left[X^l\right] = \frac{r}{p}\sum_{k=r}^{\infty} k^{l-1}\binom{k}{(r+1)-1} p^{r+1} (1-p)^{(k+1)-(r+1)}$$

Change of variable $j = k+1$:

$$\begin{aligned}\mathbf{E}\left[X^l\right] &= \frac{r}{p}\sum_{j=r+1}^{\infty} (j-1)^{l-1}\binom{j-1}{(r+1)-1} p^{r+1} (1-p)^{j-(r+1)} \\ &= \frac{r}{p}\,\mathbf{E}\left[(Y-1)^{l-1}\right]\end{aligned}$$

Computation of expectation and variance

Consequence of Proposition 13:

$$\mathbf{E}[X] = \frac{r}{p}, \qquad \mathbf{Var}(X) = \frac{r(1-p)}{p^2}$$

The Banach match problem (1)

Situation:

- Pipe smoking mathematician with 2 matchboxes
- 1 box in left hand pocket, 1 box in right hand pocket
- Each time a match is needed, selected at random
- Both boxes contain initially N matches

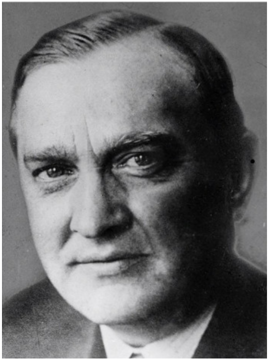
Question:

- When one box is empty,
  what is the probability that k matches are left in the other box?

Stefan Banach

Some facts about Banach:

- Lifespan: 1892-1945, in Krakow and Lviv
- Among greatest 20-th century mathematicians
- Founder of a new field
  $\hookrightarrow$ Functional Analysis
- Survived 2 world wars in tough conditions
- Then dies in 1945 from lung cancer



The Banach match problem (2)

Event: Define E_k by

(Math. discovers that rh box is empty & k matches in lh box)

Expression in terms of a negative binomial:

$$E_k = (X = N + 1 + N - k) = (X = 2N - k + 1),$$

where

$$X \sim \text{Nbin}\left(r = N + 1,\ p = \frac{1}{2}\right)$$

The Banach match problem (3)

Probability of E_k: We get

$$\mathbf{P}(E_k) = \mathbf{P}\left(X = 2N - k + 1\right) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k+1}$$

Solution to the problem:
By symmetry between left and right, we get

$$2\,\mathbf{P}(E_k) = \binom{2N-k}{N}\left(\frac{1}{2}\right)^{2N-k}$$

Hypergeometric random variable (1)

Use: Consider the experiment

- Urn containing N balls
- m white balls, $N - m$ black balls
- Sample of size n is drawn without replacement
- Set $X = \#$ white balls drawn

Then

$$X \sim \text{HypG}(n, N, m)$$

Hypergeometric random variable (2)

Notation:

$$X \sim \mathrm{HypG}(n, N, m), \quad \text{for } N \in \mathbb{N}^*, \; m, n \leq N$$

State space:

$$\{0, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}, \quad 0 \leq k \leq n$$

Expected value and variance: Set $p = \frac{m}{N}$. Then

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$$

Hypergeometric and binomial

Proposition 14.

Let

- $X \sim \text{HypG}(n, N, m)$,
- Recall that $p = \frac{m}{N}$

Hypothesis:

$$n \ll m, N, \qquad i \ll m, N$$

Then

$$\mathbf{P}(X = i) \simeq \binom{n}{i} p^i (1-p)^{n-i}$$

Proof

Expression for $\mathbf{P}(X = i)$:

$$\begin{aligned}
\mathbf{P}(X = i) &= \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \\
&= \frac{m!}{(m-i)!i!}\,\frac{(N-m)!}{(N-m-n+i)!(n-i)!}\,\frac{(N-n)!n!}{N!} \\
&= \binom{n}{i}\prod_{j=0}^{i-1}\frac{m-j}{N-j}\prod_{k=0}^{n-i-1}\frac{N-m-k}{N-i-k}
\end{aligned}$$

Approximation: If $i, j, k \ll m, N$ above, we get

$$\mathbf{P}(X = i) \simeq \binom{n}{i} p^i (1-p)^{n-i}$$

Example: electric components (1)

Situation: We have

- Lots of electric components of size 10
- We inspect 3 components per lot
  $\hookrightarrow$ Acceptance if all 3 components are non defective
- 30% of lots have 4 defective components
- 70% of lots have 1 defective component

Question:
What is the proportion of accepted lots?

Example: electric components (2)

Events: We define

- $A =$ Acceptance of a lot
- $L_1 =$ Lot with 1 defective component drawn
- $L_4 =$ Lot with 4 defective components drawn

Conditioning: We have

$$\mathbf{P}(A) = \mathbf{P}(A|\, L_1)\, \mathbf{P}(L_1) + \mathbf{P}(A|\, L_4)\, \mathbf{P}(L_4)$$

and

$$\mathbf{P}(L_1) = .7, \qquad \mathbf{P}(L_4) = .3,$$

Example: electric components (3)

Hypergeometric random variable: We check that

$$\mathbf{P}(A|\, L_1) = \mathbf{P}(X_1 = 0), \quad \text{where} \quad X_1 \sim \text{HypG}(3, 10, 1)$$

Thus

$$\mathbf{P}(A|\, L_1) = \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}}$$

Conclusion:

$$\mathbf{P}(A) = \frac{\binom{1}{0}\binom{9}{3}}{\binom{10}{3}} \times 0.7 + \frac{\binom{4}{0}\binom{6}{3}}{\binom{10}{3}} \times 0.3 = 54\%$$

Outline

Expectation of sums

Proposition 15.

Let

- **P** a probability on a sample space S
- $X_1, \ldots, X_n : S \to \mathbb{R}$ n random variables

Hypothesis: S is countable, i.e

$$S = \{s_i;\, i \geq 1\}$$

Then

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}\left[X_i\right]$$

Example: number of successes (1)

Experiment:

- n trials
- Success for i-th trial with probability p_i
- $X = \#$ of successes

Question:

Expression for $\mathbf{E}[X]$ and $\mathbf{Var}(X)$

Example: number of successes (2)

Expression for X: Let

$$X_i = \mathbf{1}_{(\text{success for } i\text{-th trial})}$$

Then

$$X = \sum_{i=1}^{n} X_i$$

Expression for $\mathbf{E}[X]$: Thanks to Proposition 15, we have

$$\mathbf{E}[X] = \sum_{i=1}^{n} p_i$$

Example: number of successes (3)

Expression for $\mathbf{E}[X^2]$: We invoke the two facts

1. $X_i^2 = X_i$
2. If $i \neq j$, $X_i X_j = \mathbf{1}_{(X_i=1, X_j=1)}$

Therefore

$$\mathbf{E}[X^2] = \sum_{i=1}^{n} \mathbf{E}[X_i^2] + \sum_{i \neq j} \mathbf{E}[X_i X_j]$$

yields

$$\mathbf{E}[X^2] = \sum_{i=1}^{n} p_i + \sum_{i \neq j} \mathbf{P}\left(X_i = 1, X_j = 1\right)$$

Example: number of successes (4)

Particular case, binomial: In this case we have

- The X_i's are independent
- $p_i = p$

New expression for $\mathbf{E}[X^2]$:

$$\mathbf{E}[X^2] = np + n(n-1)p^2$$

Expression for $\mathbf{Var}(X)$:

$$\mathbf{Var}(X) = np(1-p)$$

Example: number of successes (5)

Particular case, hypergeometric: We have

$$\begin{aligned} p_i &= \frac{m}{N} \\ \mathbf{P}\left(X_i = 1, X_j = 1\right) &= \mathbf{P}(X_i = 1)\mathbf{P}\left(X_j = 1 \middle| X_i = 1\right) \\ &= \frac{m}{N}\frac{m-1}{N-1} \end{aligned}$$

New expression for $\mathbf{E}[X^2]$:

$$\mathbf{E}[X^2] = np + n(n-1)p\,\frac{m-1}{N-1}$$

Expression for $\mathbf{Var}(X)$:

$$\mathbf{Var}(X) = np(1-p)\left(1 - \frac{n-1}{N-1}\right)$$

Outline

Continuity of the cdf

Proposition 16.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$
- F the cdf of X, i.e $F(x) = \mathbf{P}(X \leq x)$

Then the function F satisfies

1. F is a nondecreasing function
2. $\lim_{b\to\infty} F(b) = 1$
3. $\lim_{b\to-\infty} F(b) = 0$
4. F is right continuous

Proof of item 1

Inclusion property: Let $a < b$. Then

$$(X \leq a) \subset (X \leq b)$$

Consequence on probabilities:

$$\mathbf{P}(X \leq a) \leq \mathbf{P}(X \leq b)$$

Proof of item 2

Definition of an increasing sequence: Let $b_n \nearrow \infty$ and

$$E_n = (X \leq b_n)$$

Then

$$\lim_{n\to\infty} E_n = (X < \infty)$$

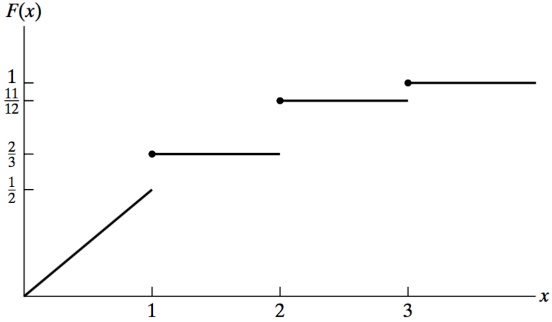
Consequence on probabilities:

$$\begin{aligned}
1 &= \mathbf{P}(X < \infty) \\
&= \mathbf{P}\left(\lim_{n\to\infty} E_n\right) \\
&= \lim_{n\to\infty} \mathbf{P}(E_n) \quad \text{(Since } n \mapsto E_n \text{ is increasing)} \\
&= \lim_{n\to\infty} F(b_n)
\end{aligned}$$

Example of cdf (1)

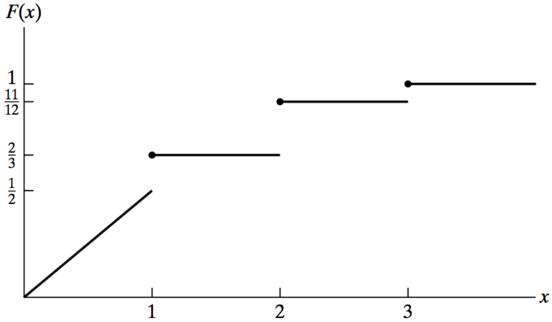
Definition of the function: We set

$$F(x) = \frac{x}{2}\,\mathbf{1}_{[0,1)}(x) + \frac{2}{3}\,\mathbf{1}_{[1,2)}(x) + \frac{11}{12}\,\mathbf{1}_{[2,3)}(x) + \mathbf{1}_{[3,\infty)}(x)$$



Example of cdf (2)

Some information read on the graph (see next page):



Example of cdf (3)

Information read on the cdf: One can check that

- $\mathbf{P}(X < 3) = \frac{11}{12}$
- $\mathbf{P}(X = 1) = \frac{1}{6}$
- $\mathbf{P}(X > \frac{1}{2}) = \frac{3}{4}$
- $\mathbf{P}(2 < X \leq 4) = \frac{1}{12}$