Aim Let $\{X_n : n \geq 1\}$ be i.i.d. r.v.

Assume $X_1 \in L^1(\Omega)$. Set $\mu = \mathbb{E}[X_1]$

Then $\overline{X}_n \xrightarrow{a.s.} \mu$

Strategy n-dependent truncation

That is

(i) Assume $X_n \geq 0$ a.s.

(ii) Set $Y_n = X_n \mathbb{1}_{(X_n \leq n)}$

We have seen

$$\mathbb{P}(X_n = Y_n \text{ except for a finite number of } n\text{'s}) = 1$$

Claim We just need to prove that

$$\bar{Y}_n \longrightarrow \mu$$

Proof of the claim

$$\bar{X}_n - \bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} (X_k - Y_k)$$

Proof of claim (ctd)

$$\overline{X}_n - \overline{Y}_n = \frac{1}{n} \sum_{k=1}^{n} (X_k - Y_k)$$

$$|\overline{X}_n - \overline{Y}_n| \leq \frac{1}{n} \sum_{k=1}^{n} \overbrace{|X_k - Y_k|}^{=0 \text{ if } k \notin A}$$

Now let $A = \{k \geq 1 ; X_k \neq Y_k\}$

We have proved that

$|A| = |A|(\omega) < \infty$ a.s.

$$\leq \underbrace{\frac{1}{n}}_{\text{fixed}} \underbrace{\sum_{k \in A} |X_k - Y_k|}_{\text{r.v.}} \xrightarrow[n \to \infty]{a.s.} 0$$

Next step: Construct a concrete subsequence for Y_n.

Recall: For LLN with $Y_1 \in L^2(\Omega)$, we had considered Y_{n_k} with $n_k = k^2$

Here we will consider $n_k = \beta_k = \lfloor \alpha^k \rfloor$ with $\alpha > 1$.

Note: For the "filling the gaps" step, we will have to take $\alpha \to 1$

Elementary inequality If $\alpha > 1$ and $\beta_k = \lfloor \alpha^k \rfloor$, then

$$\boxed{\sum_{k=m}^{\infty} \frac{1}{\beta_k^2} \leq \frac{A}{\beta_m^2}}$$

Rmk If $\hat{\beta}_k = \alpha^k$, then

$$\sum_{k=m}^{\infty} \frac{1}{\alpha^{2k}} = \frac{1}{\alpha^{2m}} \overbrace{\frac{1}{1 - 1/\alpha^2}}^{A}$$

$$\leq \frac{A}{\hat{\beta}_m^2}$$

Claim We consider $\beta_n = \lfloor \alpha^n \rfloor$.

Then

$$\boxed{\frac{1}{\beta_n}\left(S'_{\beta_n} - \underbrace{E[S'_{\beta_n}]}_{\simeq \mu \times \beta_n}\right) \xrightarrow{a.s.} 0}$$

where $S'_{\beta_n} = \sum_{k=1}^{\beta_n} Y_k$

Rmk This claim is close to

$$\overline{Y}_{\beta_n} \xrightarrow{n\to\infty} \mu$$

$\frac{1}{\beta_n}\left(S'_{\beta_n} - \underbrace{E[S'_{\beta_n}]}\right) \xrightarrow{a.s.} 0$

<u>Method</u>: Through B-C. We set

$$B_n(\varepsilon) = \left(\frac{1}{\beta_n}\left|S'_{\beta_n} - E[S'_{\beta_n}]\right| > \varepsilon\right)$$

It is enough to prove that

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n(\varepsilon)) < \infty \quad \forall \varepsilon$$

<u>Advantage</u>: Now every $Y_k \in L^2$.
We can apply Chevychell

$$B_n(\varepsilon) = \left(\frac{1}{\beta_n} |S'_{\beta_n} - \mathbb{E}[S'_{\beta_n}]| > \varepsilon \right)$$

$$\boxed{\mathbb{P}(B_n(\varepsilon))} = \mathbb{P}\Big(\underbrace{\frac{1}{\beta_n} |S'_{\beta_n} - \mathbb{E}[S'_{\beta_n}]|}_{|Z_n|} > \varepsilon \Big)$$

Chebychell

$$\leq \frac{\mathbb{E}[|Z_n|^2]}{\varepsilon^2}$$

$$= \frac{1}{\beta_n^2 \varepsilon^2} \mathbb{E}\left[|S'_{\beta_n} - \mathbb{E}[S'_{\beta_n}]|^2 \right]$$

$$= \frac{1}{\beta_n^2 \varepsilon^2} \mathrm{Var}(S'_{\beta_n})$$

$$\boxed{= \frac{1}{\beta_n^2 \varepsilon^2} \mathrm{Var}\left(\sum_{k=1}^{\beta_n} Y_k \right)}$$

$$P(B_n(\varepsilon)) \leq \frac{1}{\beta_n^2 \varepsilon^2} \operatorname{Var}\left(\sum_{k=1}^{\beta_n} Y_k\right)$$

Question: we had X_n's $\perp\!\!\!\perp$ and $Y_n = X_n \mathbb{1}(X_n \leq n)$. Do we have Y_n's $\perp\!\!\!\perp$?

$\hookrightarrow$ $Y_n = \varphi_n(X_n)$ with $\varphi_n(x) = x\,\mathbb{1}(x \leq n)$

Rule: If $Z_1 \perp\!\!\!\perp Z_2$ and $\varphi_1\ \varphi_2$ are 2 meas. functions, then

$$\varphi_1(Z_1) \perp\!\!\!\perp \varphi_2(Z_2)$$

This can be extended to a countable family $(Z_k)_{k\geq 1}$

$$\boxed{\mathbb{P}(B_n(\varepsilon))} \leq \frac{1}{\beta_n^2 \varepsilon^2} \operatorname{Var}\left(\sum_{k=1}^{\beta_n} Y_k\right)$$

$$= \frac{1}{\beta_n^2 \varepsilon^2} \sum_{k=1}^{\beta_n} \operatorname{Var}(Y_k)$$

$$\Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(B_n(\varepsilon)) = \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \sum_{k=1}^{\beta_n} \operatorname{Var}(Y_k)$$

Fubini

$$= \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \operatorname{Var}(Y_k) \sum_{n;\, \beta_n \geq k} \frac{1}{\beta_n^2}$$

elementary ineq

$$\leq c_1 \frac{1}{\varepsilon^2} \sum_{k=1}^{\infty} \underbrace{\operatorname{Var}(Y_k)}_{\leq E[Y_k^2]} \frac{1}{k^2} \leq \boxed{\frac{c_1}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{E[Y_k^2]}{k^2}}$$

$Y_k = X_k \mathbb{1}(X_k < k) \Rightarrow Y_k \le k$ $\quad Y_k = X_k$ if $Y_k \in (0, k]$

Summary We have seen

$$\sum_{k=1}^{\infty} \frac{E[Y_k^2]}{k^2} < \infty \Rightarrow \sum_{n=1}^{\infty} P(B_n(\varepsilon)) < \infty$$

$$\boxed{\sum_{k=1}^{\infty} \frac{E[Y_k^2]}{k^2}} = \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} E[Y_k^2 \mathbb{1}\overbrace{(j-1 < Y_k \le j)}^{B_{kj}}]$$

$$\le \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 P(j-1 \le Y_k \le j)$$

$\downarrow j \le k$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 P(j-1 \le X_k \le j)$$

i.d

$$\boxed{= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 P(j-1 \le X_1 \le j)}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} \mathbb{E}[X_k^2]}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 \, \mathbb{P}(j-1 \leq X_1 \leq j)$$

Fubini

$$= \sum_{j=1}^{\infty} j^2 \, \mathbb{P}(j-1 \leq X_1 \leq j) \sum_{k=j}^{\infty} \frac{1}{k^2} \quad \leq \frac{c_2}{j} \text{ (calculus)}$$

$$\leq c_2 \sum_{j=1}^{\infty} j \, \mathbb{P}(j-1 \leq X_1 \leq j)$$

$$\leq c_2 \sum_{j=1}^{\infty} j \, \mathbb{P}(X_1 \geq j-1)$$

Prob 3

$X_1 \in \mathcal{L}^1(\Omega)$

$$\boxed{\leq c_2 (1 + \mathbb{E}[X_1]) < \infty}$$

Proof of Theorem 16 (2)

Proof of claim (4): We have

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \sum_{n=1}^{\infty} \mathbf{P}(X_n \geq n) \leq \mathbf{E}[X_1] < \infty$$

Thus (4) holds thanks to Borel-Cantelli

Proof of Theorem 16 (3)

Reduction of the proof: According to (4), we have

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{\text{a.s}} 0$$

Hence we just need to show

$$\bar{Y}_n \xrightarrow{\text{a.s}} \mu$$

Proof of Theorem 16 (4)

Elementary relation: Let $\alpha > 1$ and $\beta_k = \lfloor \alpha^k \rfloor$.
Then there exists $A > 0$ such that

$$\sum_{k=m}^{\infty} \frac{1}{\beta_k^2} \leq \frac{A}{\beta_m^2} \tag{5}$$

Brief proof of (5): Stems from

$$\beta_k \asymp \alpha^k, \quad \text{for large } k\text{'s}$$

Proof of Theorem 16 (5)

Claim 2 about the truncation: Write $S'_n = \sum_{k=1}^{n} Y_k$. Then

$$\frac{1}{\beta_n}\left(S'_{\beta_n} - \mathbf{E}\left[S'_{\beta_n}\right]\right) \xrightarrow{\text{a.s}} 0 \tag{6}$$

Proof of Theorem 16 (6)

Proof of (6): For $\varepsilon > 0$, set

$$B_n(\varepsilon) = \left(\frac{1}{\beta_n} \left| S'_{\beta_n} - \mathbf{E}\left[S'_{\beta_n} \right] \right| > \varepsilon \right)$$

Then the following yields (6) by Borel-Cantelli:

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbf{P}\left(B_n(\varepsilon) \right) &\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \mathbf{Var}\left(S'_{\beta_n} \right) \\
&\leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\beta_n^2} \sum_{k=1}^{\beta_n} \mathbf{Var}\left(Y_k \right) \\
&\leq \frac{A}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{E}\left[Y_k^2 \right] \overset{\text{Claim 3}}{<} \infty
\end{aligned}$$

Proof of Theorem 16 (7)

Proof of Claim 3: This is where we use the truncation,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} \mathbf{E}\left[Y_k^2\right] &= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} \mathbf{E}\left[Y_k^2 \mathbf{1}_{B_{kj}}\right] \quad (B_{kj} = (j-1 \leq X_k < j)) \\
&\leq \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 \mathbf{P}\left(B_{kj}\right) \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{j=1}^{k} j^2 \mathbf{P}\left(B_{1j}\right) \\
&= \sum_{j=1}^{\infty} j^2 \mathbf{P}\left(B_{1j}\right) \sum_{k=j}^{\infty} \frac{1}{k^2} \\
&\lesssim \sum_{j=1}^{\infty} j \mathbf{P}\left(B_{1j}\right) \lesssim 1 + \sum_{j=1}^{\infty} (j-1) \mathbf{P}\left(B_{1j}\right) \\
&\lesssim 1 + \mathbf{E}[X_1] < \infty
\end{aligned}$$

Proof of Theorem 16 (8) $\frac{1}{\beta_n}(S'_{\beta_n} - \mathbf{E}[S'_{\beta_n}]) \xrightarrow{a.s.} 0$

From (6) to the theorem: The missing steps are

1. We have $\mathbf{E}[Y_n] \to \mu$
   $\hookrightarrow$ by monotone convergence

   $E[Y_n] = \mathbf{E}[X_n \mathbb{1}_{(X_n < n)}]$
   $= \mathbf{E}[X_1 \mathbb{1}_{(X_1 \leq n)}]$
   $\to \mu$

2. Fill the gaps between β_n's
   $\hookrightarrow$ Similar to Proposition 15

   $\beta_n = \lfloor \alpha^n \rfloor$, with $\alpha \to 1$

3. Signed sequence, also like in Proposition 15:
   1. Write $X_n = X_n^+ - X_n^-$
   2. Apply positive sequence case to both X_n^+ and X_n^-
   3. This is allowed since $X_n^{\pm}$ i.i.d with $\mathbf{E}[X_1^{\pm}] < \infty$

Conclusion: We have

$$X_1 \in L^1 \quad \Longrightarrow \quad \bar{X}_n \xrightarrow{\text{a.s}} \mu$$

Proof of Theorem 16 (9)

Aim $\bar{X}_n \to \mu \Rightarrow X_1 \in L^1(\Omega)$

Converse result: We have

$$\mathbf{P}\left(\limsup \frac{|X_n|}{n} > 1\right) = 0$$

$$\bar{X}_n \xrightarrow{\text{a.s}} \mu \quad \overset{\text{results on series}}{\Longrightarrow} \quad \frac{X_n}{n} \xrightarrow{\text{a.s}} 0$$

$$\overset{\text{reversed Borel-C}}{\Longrightarrow} \quad \sum_{n=1}^{\infty} \mathbf{P}\left(|X_n| \geq n\right) < \infty$$

$$\overset{\text{i.i.d Hyp}}{\Longrightarrow} \quad \sum_{n=1}^{\infty} \mathbf{P}\left(|X_1| \geq n\right) < \infty$$

Hence

$$\mathbf{E}[|X_1|] \overset{\text{Problem 4.14.3}}{\leq} 1 + \sum_{n=1}^{\infty} \mathbf{P}\left(|X_1| \geq n\right) < \infty$$