Cauchy-Schwarz inequality

so that $XY \in L^1(\Omega)$ and thus
$\mathbf{E}[XY|\mathcal{F}]$ well defined

Proposition 17.

Let $X, Y \in L^2(\Omega)$. Then

$$\mathbf{E}^2[X\,Y|\mathcal{F}] \leqslant \mathbf{E}[X^2|\mathcal{F}]\,\mathbf{E}[Y^2|\mathcal{F}] \quad \text{a.s.}$$

Proof

$\forall \theta \in \mathbb{R}$, $(X+\theta Y)^2 \geq 0$ a.s. Thus

$\left(\mathbb{E}[(X+\theta Y)^2 \mid \mathcal{F}] \geq 0 \quad \text{a.s} \right)$

We would like to say

True ?

$\left(\mathbb{E}[(X+\theta Y)^2 \mid \mathcal{F}] \geq 0 \quad \forall \theta \in \mathbb{R} \right)$ a.s.

Define $A_\theta = \left(\mathbb{E}[(X+\theta Y)^2 | \mathcal{F}] \geq 0 \right)$

We know that $\mathbb{P}(A_\theta) = 1$

We would like to have

uncountable $\cap$!

$$\mathbb{P}\left(\mathbb{E}[(X+\theta Y)^2 | \mathcal{F}] \geq 0 \ \forall \theta \right) = 1$$

$$\Leftrightarrow \mathbb{P}\left(\bigcap_{\theta \in \mathbb{R}} A_\theta \right) = 1$$

What we can write instead is

$$\boxed{\mathbb{P}\left(\bigcap_{\theta \in \mathbb{Q}} A_\theta \right) = 1}$$

$Z \geq 0$ a.s. $\Leftrightarrow$ $\mathbb{P}(Z \geq 0) = 1$

We get

$$\left(\mathbb{E}[(X+\theta Y)^2 \mid \mathcal{F}] \geq 0 \quad \forall \theta \in \mathbb{Q} \right) \text{ a.s.}$$

This means $\forall \theta \in \mathbb{Q}$, (linearity of $\mathbb{E}[\cdot \mid \mathcal{F}]$)

$$\mathbb{E}[Y^2 \mid \mathcal{F}]\,\theta^2 + 2\theta\, \mathbb{E}[XY \mid \mathcal{F}] + \mathbb{E}[X^2 \mid \mathcal{F}] \geq 0$$

By continuity of polynomial, the discriminant of this polynomial must be ≤ 0. Thus

$$4 \left(\mathbb{E}[XY \mid \mathcal{F}] \right)^2 - 4\, \mathbb{E}[Y^2 \mid \mathcal{F}]\, \mathbb{E}[X^2 \mid \mathcal{F}] \leq 0$$

$$\Rightarrow \boxed{\left(\mathbb{E}[XY \mid \mathcal{F}] \right)^2 \leq \mathbb{E}[X^2 \mid \mathcal{F}]\, \mathbb{E}[Y^2 \mid \mathcal{F}] \text{ a.s.}}$$

Proof of Cauchy-Schwarz (1)

A family positive random variables:
For all $\theta \in \mathbb{R}$, we have

$$\mathbf{E}[(X + \theta Y)^2 | \mathcal{F}] \geqslant 0 \quad \text{a.s.}$$

Thus almost surely we have: for all $\theta \in \mathbb{Q}$,

$$\mathbf{E}[(X + \theta Y)^2 | \mathcal{F}] \geqslant 0,$$

Proof of Cauchy-Schwarz (2)

Expansion: For all $\theta \in \mathbb{Q}$

$$\mathbf{E}[Y^2|\mathcal{F}]\theta^2 + 2\mathbf{E}[XY|\mathcal{F}]\theta + \mathbf{E}[X^2|\mathcal{F}] \geqslant 0.$$

Recall: If a polynomial satisfies $a\theta^2 + b\theta + c \geqslant 0$ for all $\theta \in \mathbb{Q}$
$\hookrightarrow$ then we have $b^2 - 4ac \leqslant 0$

Application: Almost surely, we have

$$E^2[XY|\mathcal{F}] - \mathbf{E}[X^2|\mathcal{F}]\mathbf{E}[Y^2|\mathcal{F}] \leqslant 0.$$

Jensen's inequality

$\varphi : \mathbb{R} \to \mathbb{R}$ convex : If $\alpha \in (0,1)$, $x_1, x_2 \in \mathbb{R}$

$$\alpha \varphi(x_1) + (1-\alpha)\varphi(x_2) \geq \varphi(\alpha x_1 + (1-\alpha) x_2)$$

Proposition 18.

Let $X \in L^1(\Omega)$, and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(X) \in L^1(\Omega)$ and φ convex. Then

$$\varphi(\mathbf{E}[X|\mathcal{F}]) \leqslant \mathbf{E}[\varphi(X)|\mathcal{F}] \quad \text{a.s.}$$

Typical example of convex φ :

$\varphi(x) = |x|^p$, $p \geq 1$

Contraction in $L^p(\Omega)$

$$\|Z\|_{L^p(\Omega)} = \left(\mathbf{E}[|Z|^p]\right)^{1/p}$$

Proposition 19.

The conditional expectation is a

contraction in $L^p(\Omega)$ for all $p \geqslant 1$

i.e $\quad \|\mathbf{E}[X|\mathcal{F}]\|_{L^p(\Omega)} \leq \|X\|_{L^p(\Omega)}$

or $\quad \mathbf{E}[|\mathbf{E}[X|\mathcal{F}]|^p] \leq \mathbf{E}[|X|^p]$

Aim $\mathbb{E}[\,|\mathbb{E}[X|\mathcal{F}]|^p] \leq \mathbb{E}[|X|^p]$

Proof: By Jensen for $\varphi(x) = |x|^p$

$$\mathbb{E}\{|\mathbb{E}[X|\mathcal{F}]|^p\} \leq \mathbb{E}\{\mathbb{E}[|X|^p|\mathcal{F}]\}$$

$$= \mathbb{E}[|X|^p]$$

$$\left(\mathbb{E}\{\mathbb{E}[Z|\mathcal{F}]\} = \mathbb{E}[Z] \right)$$

Proof of contraction in L^p

Application of Jensen's inequality: We have

$$X \in L^p(\Omega) \Rightarrow \mathbf{E}[X|\mathcal{F}] \in L^p(\Omega)$$

and

$$|\mathbf{E}[X|\mathcal{F}]|^p \leq \mathbf{E}[|X|^p|\mathcal{F}] \quad \Longrightarrow \quad \mathbf{E}\left\{|\mathbf{E}[X|\mathcal{F}]|^p\right\} \leqslant \mathbf{E}[|X|^p]$$

Successive conditionings

Theorem 20.

If $A \in \mathcal{F}_1$, then $A \in \mathcal{F}_2$

Let

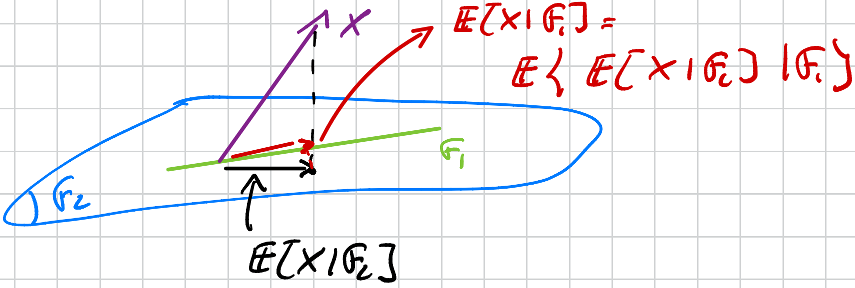
- Two σ-algebras $\mathcal{F}_1 \subset \mathcal{F}_2$.
- $X \in L^1(\Omega)$.

Then

$$\mathbf{E}\left\{\mathbf{E}[X|\mathcal{F}_1]|\mathcal{F}_2\right\} = \mathbf{E}[X|\mathcal{F}_1] \quad (2)$$

$$\mathbf{E}\left\{\mathbf{E}[X|\mathcal{F}_2]|\mathcal{F}_1\right\} = \mathbf{E}[X|\mathcal{F}_1]. \quad (3)$$

Illustration



⚠ This picture is misleading.
$\mathbb{F}_1, \mathbb{F}_2$ are not vector spaces

Proof Set $Z_1 = \mathbb{E}[X \mid \mathcal{F}_1]$
$Z_2 = \mathbb{E}[X \mid \mathcal{F}_2]$

(1) We wish to prove

$$\mathbb{E}[Z_1 \mid \mathcal{F}_2] = Z_1$$

$$\mathbb{E}\{\mathbb{E}[X \mid \mathcal{F}_1] \mid \mathcal{F}_2\} = \mathbb{E}[X \mid \mathcal{F}_1]$$

We have seen $Y \in \mathcal{F} \Rightarrow \mathbb{E}[Y \mid \mathcal{F}] = Y$

Here $Z_1 = \mathbb{E}[X \mid \mathcal{F}_1]$ is $\mathcal{F}_1$-meas.

$\Rightarrow$ Z_1 is $\mathcal{F}_2$-meas.

$$\Rightarrow \boxed{\mathbb{E}[Z_1 \mid \mathcal{F}_2] = Z_1}$$

(2) We wish to see $\mathbb{E}[Z_2 \mid \mathcal{F}_1] = Z_1$

$\left(\mathbb{E}\{ \mathbb{E}[X \mid \mathcal{F}_2] \mid \mathcal{F}_1 \} = \mathbb{E}[X \mid \mathcal{F}_1] \right)$

We verify (i) and (ii) for Z_1:

(i) $Z_1 \in \mathcal{F}_1$, true since $Z_1 = \mathbb{E}[X \mid \mathcal{F}_1]$

(ii) Take $A_1 \in \mathcal{F}_1$. We have

$$\begin{array}{ccc} \mathbb{E}[Z_1 \mathbb{1}_{A_1}] & \overset{Z_1 = \mathbb{E}[X \mid \mathcal{F}_1]}{=} & \mathbb{E}[X \mathbb{1}_{A_1}] \\ \parallel & & \\ \mathbb{E}[Z_2 \mathbb{1}_{A_1}] & \overset{A_1 \in \mathcal{F}_2}{=} & \mathbb{E}[X \mathbb{1}_{A_1}] \end{array}$$

Conclusion: $\boxed{\mathbb{E}[Z_2 \mid \mathcal{F}_1] = Z_1}$

Proof

Proof of (2): We set $Z \equiv \mathbf{E}[X|\mathcal{F}_1]$. Then

$$Z \in \mathcal{F}_1 \subset \mathcal{F}_2.$$

According to Example 1, we have $\mathbf{E}[Z|\mathcal{F}_2] = Z$, i.e. (2).

Proof of (3): We set $U = \mathbf{E}[X|\mathcal{F}_2]$.
$\hookrightarrow$ We will show that $\mathbf{E}[U|\mathcal{F}_1] = Z$, via (i) and (ii) of Definition 6.

(i) $Z \in \mathcal{F}_1$.

(ii) If $A \in \mathcal{F}_1$, we have $A \in \mathcal{F}_1 \subset \mathcal{F}_2$, and thus

$$\mathbf{E}[Z\mathbf{1}_A] = \mathbf{E}[X\mathbf{1}_A] = \mathbf{E}[U\mathbf{1}_A].$$

Conditional expectation for products

Recipe for $\mathbf{E}[\cdot|\mathcal{F}]$: • freeze what you know
• average what you don't know

Theorem 21.

Let $X, Y \in L^2(\Omega)$, such that $X \in \mathcal{F}$. Then

$$\mathbf{E}[X\,Y|\mathcal{F}] = X\,\mathbf{E}[Y|\mathcal{F}].$$

Proof: We use a 4 steps methodology

(1) X = indicator

(2) X = linear combination of indicators

(3) $X \geq 0$, $X_n \nearrow X$

(4) X with arbitrary sign

Proof (I) $X = \mathbb{1}_B$ with $B \in \mathcal{F}$.

Claim: $Z = \mathbb{1}_B \, \mathbb{E}[Y \mid \mathcal{F}]$ is s.t.

$Z = \mathbb{E}[\mathbb{1}_B Y \mid \mathcal{F}]$

Indeed (i) $Z \in \mathcal{F}$, as product of r.v. in $\mathcal{F}$

(ii) Take $A \in \mathcal{F}$. Then

$$\mathbb{E}[Z \, \mathbb{1}_A] = \mathbb{E}[\mathbb{1}_B \, \mathbb{E}[Y \mid \mathcal{F}] \, \mathbb{1}_A]$$

$$= \mathbb{E}[\mathbb{E}[Y \mid \mathcal{F}] \, \mathbb{1}_{\overset{\in \mathcal{F}}{\frown}\atop A \cap B}] = \mathbb{E}[Y \, \mathbb{1}_{A \cap B}]$$

$$= \mathbb{E}[\mathbb{1}_B Y \, \mathbb{1}_A] = \mathbb{E}[X Y \, \mathbb{1}_A]$$

Conclusion: we have seen

$$\mathbb{E}[Z \, \mathbb{1}_A] = \mathbb{E}[XY \, \mathbb{1}_A] \quad \forall A \in \mathcal{F}$$

Thus $\boxed{Z = \mathbb{1}_B \, \mathbb{E}[Y \mid \mathcal{F}] \text{ is also } \mathbb{E}[XY \mid \mathcal{F}]}$

(2) Take $B_1, \dots, B_n \in \mathcal{F}$
$\alpha_1, \dots, \alpha_n \in \mathbb{R}$

Claim: $\mathbb{E}\left[\left(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}\right) Y \mid \mathcal{F}\right]$

$$= \left(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}\right) \mathbb{E}[Y \mid \mathcal{F}]$$

Proof of claim

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}\Big) Y \,\Big|\, \mathcal{F}\Big]$$

$$= \mathbb{E}\Big[\sum_{i=1}^{n} (\alpha_i \mathbb{1}_{B_i} Y) \,\Big|\, \mathcal{F}\Big]$$

linearity

$$= \sum_{i=1}^{n} \alpha_i \, \mathbb{E}[\mathbb{1}_{B_i} Y \mid \mathcal{F}]$$

Step 1

$$= \Big(\sum_{i=1}^{n} \alpha_i \mathbb{1}_{B_i}\Big) \mathbb{E}[Y \mid \mathcal{F}]$$

Step 3 Take $X, Y \geq 0$.

Then $\exists \{X_n ; n \geq 1\}$ such that each $X_n = \sum \alpha_i \mathbb{1}_{B_i}$ and

$$X_n \nearrow X \quad \text{a.s.}$$

Thus $\forall n$

$$\begin{array}{ccc} \mathbb{E}[X_n Y \mid \mathcal{F}] & = & X_n \, \mathbb{E}[Y \mid \mathcal{F}] \\ \Big\downarrow {\scriptstyle n\to\infty} & & \Big\downarrow {\scriptstyle n\to\infty} \\ \mathbb{E}[XY \mid \mathcal{F}] & = & X \, \mathbb{E}[Y \mid \mathcal{F}] \end{array}$$

Beppo-Levi for $\mathbb{E}[\cdot \mid \mathcal{F}]$

Proof

Step 1: Assume $X = \mathbf{1}_B$, with $B \in \mathcal{F}$
We check (i) and (ii) of Definition 6.

(i) We have $\mathbf{1}_B \mathbf{E}[Y|\mathcal{F}] \in \mathcal{F}$.

(ii) For $A \in \mathcal{F}$, we have

$$\begin{aligned}
\mathbf{E}\left\{(\mathbf{1}_B \mathbf{E}[Y|\mathcal{F}])\, \mathbf{1}_A\right\} &= \mathbf{E}\left\{\mathbf{E}[Y|\mathcal{F}]\, \mathbf{1}_{A\cap B}\right\} \\
&= \mathbf{E}[Y\, \mathbf{1}_{A\cap B}] \\
&= \mathbf{E}[(\mathbf{1}_B Y)\, \mathbf{1}_A],
\end{aligned}$$

and thus

$$\mathbf{1}_B\, \mathbf{E}[Y|\mathcal{F}] = \mathbf{E}[\mathbf{1}_B\, Y|\mathcal{F}].$$

Proof (2)

Step 2: If X is of the form

$$X = \sum_{i \leqslant n} \alpha_i \mathbf{1}_{B_i},$$

with $\alpha_i \in \mathbb{R}$ and $B_i \in \mathcal{F}$, then, by linearity we also get

$$\mathbf{E}[XY|\mathcal{F}] = X\,\mathbf{E}[Y|\mathcal{F}].$$

Step 3: If $X, Y \geqslant 0$

$\hookrightarrow$ There exists a sequence $\{X_n; n \geqslant 1\}$ of simple random variables such that

$$X_n \nearrow X.$$

Then applying the monotone convergence we end up with:

$$\mathbf{E}[XY|\mathcal{F}] = X\,\mathbf{E}[Y|\mathcal{F}].$$

Proof (3)

Step 4: General case $X \in L^2$

$\hookrightarrow$ Decompose $X = X^+ - X^-$ and $Y = Y^+ - Y^-$, which gives

$$\mathbf{E}[XY|\mathcal{F}] = X\mathbf{E}[Y|\mathcal{F}]$$

by linearity.