Rmk In general

X, Y decorrelated $\overset{\Leftarrow}{\not\Rightarrow}$ $X \perp\!\!\!\perp Y$

i.e if X Y are centered

$$\mathbb{E}[XY] = 0 \not\Rightarrow X \perp\!\!\!\perp Y$$

However if (X, Y) is a Gauss. vector

X, Y decorrelated $\Rightarrow$ $X \perp\!\!\!\perp Y$

Another way to state this property:

(X,Y) has a covariance matrix which is diagonal

If (X,Y) Gauss vector

$\Rightarrow \quad X \perp\!\!\!\perp Y$

$$\mathrm{cov}(X,Y) = \begin{pmatrix} V(X) & E[XY] \\ E[XY] & V(Y) \end{pmatrix}$$

Example of (X, Y) with

$\mathbb{E}[XY] = 0$, but $X \not\perp\!\!\!\perp Y$

Take

$X \sim \mathcal{N}(0,1)$ $\quad \varepsilon \perp\!\!\!\perp X$ $\quad \mathbb{P}(\varepsilon = \pm 1) = \frac{1}{2}$

$Y = \varepsilon X$

Then $\mathbb{E}[XY] = \mathbb{E}[\varepsilon \overset{\perp\!\!\!\perp}{\leftrightarrow} X^2]$

$= \mathbb{E}[\varepsilon]\, \mathbb{E}[X^2] = 0$

But $\boxed{X \not\perp\!\!\!\perp Y}$ $\longrightarrow$ check $\mathbb{P}(X \in A, Y \in B)$

$\neq \mathbb{P}(X \in A)\, \mathbb{P}(Y \in B)$

Back to our example: we have seen that

$$\mathbb{E}[(X-\alpha Y)\,Y] = 0 \quad \text{(decorrelation)}$$

In addition $(X-\alpha Y, Y)$ is a Gauss. vector

Thus $\quad (X-\alpha Y) \perp\!\!\!\perp Y$

$$\Rightarrow \quad (X-\alpha Y) \perp\!\!\!\perp \varphi(Y)$$

$$\Rightarrow \quad \mathbb{E}[(X-\alpha Y)\,\varphi(Y)] = 0 \quad \text{Hyp: } \varphi(Y) \in L^2$$

$$\Rightarrow \quad \boxed{\alpha Y = \mathbb{E}[X \mid Y]}$$

Rmk

If X, Y are centered, $X \perp\!\!\!\perp Y$, then

$$\mathbb{E}[XY] \overset{\perp\!\!\!\perp}{=} \mathbb{E}[X]\,\mathbb{E}[Y] = 0$$

Proof

Step 1: We look for α such that

$$Z = X - \alpha Y \quad \Longrightarrow \quad Z \perp\!\!\!\perp Y.$$

Recall: If (Z, Y) is a Gaussian vector
$\hookrightarrow Z \perp\!\!\!\perp Y$ iff $\mathrm{cov}(Z, Y) = 0$

Application: $\mathrm{cov}(Z, Y) = \mathbf{E}[Z\, Y]$. Thus

$$\mathrm{cov}(Z, Y) = \mathbf{E}[(X - \alpha Y)\, Y] = \mathbf{E}[X\, Y] - \alpha V(Y),$$

et

$$\mathrm{cov}(Z, Y) = 0 \quad \text{iff} \quad \alpha = \frac{\mathbf{E}[XY]}{V(Y)}.$$

Proof (2)

Step 2: We invoke (i) in the definition of π.
$\hookrightarrow$ Let $V \in L^2(\sigma(Y))$. Then

$$Y \perp\!\!\!\perp (X - \alpha Y) \quad \Longrightarrow \quad V \perp\!\!\!\perp (X - \alpha Y)$$

and

$$\mathbf{E}[(X - \alpha Y)\, V] = \mathbf{E}[X - \alpha Y]\, \mathbf{E}[V] = 0.$$

Thus

$$\alpha Y = \pi_{\sigma(Y)}(X) = \mathbf{E}[X|\, Y].$$

Outline

1 Definition
- Baby conditional distributions: discrete case
- Baby conditional distributions: continuous case
- Definition with measure theory

2 Examples

3 Existence and uniqueness

4 Conditional expectation: properties

5 Conditional expectation as a projection

6 Conditional regular laws
- Probability laws and expectations
- Definition of the CRL

Aim of this section

Recall: We have seen that if

- $X \sim \mathcal{P}(\lambda_1)$, $Y \sim \mathcal{P}(\lambda_2)$
- $X \perp\!\!\!\perp Y$
- $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$,

then

$$\mathcal{L}\left(X \middle| X + Y = n\right) = \text{Bin}(n, p)$$

Question:
How to translate this
$\hookrightarrow$ to the non-baby conditional expectation language?

Rmk In the Poisson case

$\mathcal{L}(X \mid \overbrace{X+Y}^{S} = n) = \text{Bin}(n, p)$

translation $\rightarrow$ $\mathcal{L}(X \mid S) = \text{Bin}(S(\omega), p)$

2 Problems

(i) We have computed $\mathbb{E}[\varphi(x) \mid S]$

$\rightarrow$ Find a way to go from $\mathbb{E}$ to $\mathcal{L}$

(ii) what is a random probability measure ?

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Characterizing r.v by expected values

Notation:

$C_b(\mathbb{R}) \equiv$ set of continuous and bounded functions on $\mathbb{R}$.

Theorem 25.

Let X be a r.v. We assume that

$$\mathbf{E}[\varphi(X)] = \int_{\mathbb{R}} \varphi(x)\, f(x)\, dx, \quad \text{for all functions } \varphi \in C_b(\mathbb{R}).$$

Then X is continuous, with density f.

Here, we characterize $\mathcal{L}(X)$ through
$\{E[\varphi(X)];\ \varphi \in C_b(\mathbb{R})\}$

Application: change of variable

Standard method: $\mathbb{P}(Y \le y) = \mathbb{P}(h(X) \le y)$

$\to$ we get the cdf of Y (F_Y)

then $f_Y = F_Y'$

Problem: Let

- X random variable with density f.
- Set $Y = h(X)$ with $h : \mathbb{R} \to \mathbb{R}$.

We wish to find the density of Y.

Application: change of variable (2)

$Y = h(X)$
X has density f

Recipe: One proceeds as follows

1. For $\varphi \in C_b(\mathbb{R})$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(h(X))] = \int_{\mathbb{R}} \varphi(\underbrace{h(x)}_{=y}) \, f(x) \, dx.$$

2. Change variables $y = h(x)$ in the integral.
After some elementary computations we get

$$\mathbf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y) \, g(y) \, dy.$$

3. This characterizes Y, which admits a density g

Example: normal r.v and linear transformations

Proposition 26.

Let

- $X \sim \mathcal{N}(0,1)$
- $\mu \in \mathbb{R}$ and $\sigma > 0$
- Set $Y = \sigma X + \mu$

Then

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

Proof $X \sim \mathcal{N}(0,1)$, $Y = \sigma X + \mu$.

Take $\varphi \in C_b(\mathbb{R})$. Then

$$\mathbb{E}[\varphi(Y)] = \mathbb{E}[\varphi(\sigma X + \mu)]$$

$$= \int_{\mathbb{R}} \varphi(\sigma x + \mu) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

CV: $y = \sigma x + \mu \Rightarrow x = \frac{y-\mu}{\sigma} \quad dx = \frac{dy}{\sigma}$

$$= \int_{\mathbb{R}} \varphi(y) \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy \quad \forall \varphi \in C_b(\mathbb{R})$$

$\Rightarrow$ density of Y: $\dfrac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{(2\pi\sigma^2)^{\frac{1}{2}}} \Rightarrow \boxed{Y \sim \mathcal{N}(\mu, \sigma^2)}$

Proof

Recipe, item 1: for $\varphi \in C_b(\mathbb{R})$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(\sigma X + \mu)] = \int_{\mathbb{R}} \varphi(\sigma x + \mu) \, \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx .$$

Recipe, item 2: Change of variable: $y = \sigma x + \mu$:

$$\mathbf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y) \, g(y) \, dx, \quad \text{with} \quad g(y) = \frac{e^{-(y-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}} .$$

Recipe, item 3:
Y is continuous with density g, therefore $Y \sim \mathcal{N}(\mu, \sigma^2)$.

Characterizing r.v by expected values (ctd)

Theorem 27.

Let $X : \Omega \to \mathbb{R}$ be a r.v. Then

$$\{\mathbf{E}[\varphi(X)];\ \varphi \in C_b(\mathbb{R})\} \text{ characterizes the law of } X$$

$$\{\mathbb{E}[e^{iuX}];\ u \in \mathbb{R}\}$$

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CRL

Definition 28.

Let

- $(\Omega, \mathcal{F}, P)$ a probability space
- $(S, \mathcal{S})$ a measurable space of the form $\mathbb{R}^d, \mathbb{Z}^d$
- $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ a random variable in $L^1(\Omega)$
- $\mathcal{G}$ a σ-algebra such that $\mathcal{G} \subset \mathcal{F}$.

We say that $\mu : \Omega \times \mathcal{S} \to [0, 1]$ is a Conditional regular law of X given $\mathcal{G}$ if

(i) For all $f \in C_b(S)$, the map $\omega \mapsto \mu(\omega, f)$ is a random variable, equal to $\mathbf{E}[f(X)|\, \mathcal{G}]$ a.s.

(ii) ω-a.s. $f \mapsto \mu(\omega, f)$ is a probability measure on $(S, \mathcal{S})$.

Discrete example

Poisson law case: Let

- $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$
- $X \perp\!\!\!\perp Y$

We set $S = X + Y$.

Then

$$\text{CRL of } X \text{ given } S \text{ is } \mathrm{Bin}(S, p), \text{ with } p = \frac{\lambda}{\lambda+\mu}$$

Proof: We know that

$$\mathcal{L}(X \mid S = n) = \mathrm{Bin}(n, p)$$

We have also seen (for $\varphi \in C_b(\mathbb{R})$)

$\mathbb{E}[\varphi(X) \mid S]$ $\qquad \mathcal{G} = \sigma(S)$

$$= \sum_{k=0}^{S} \binom{S}{k} p^k (1-p)^{S-k}$$

Claim 1: (i) in the definition is satisfied, ie for a fixed φ,

$\mathbb{E}[\varphi(X) \mid S]$ is a r.v.

Justification

$S : \Omega \longrightarrow \mathbb{R}$ measurable
$\omega \longmapsto S(\omega)$

$$\mathbb{E}[\varphi(X) \mid S]$$

$$= \sum_{k=0}^{S} \binom{S}{k} p^k (1-p)^{S-k}$$

$$= \sum_{k=0}^{\infty} \underbrace{\binom{S}{k}}_{\text{polynomial in } S} p^k \underbrace{(1-p)^{S-k}}_{\text{exp function}} \underbrace{\mathbb{1}(S \geq k)}_{\text{indicator function}}$$

This is a r.v (measurable)

Justification of (ii). If we fix ω, we have that

$S = X + Y \sim \mathcal{P}(\lambda_1 + \lambda_2)$

$$S(\omega) \in \mathbb{N}$$

and $\bar{\mathcal{L}}(X \mid S)[(\omega)$

is $\mathrm{Bin}(S(\omega), p)$

This is a probability distribution

Conclusion: we can write safely

$$\boxed{\mathcal{L}(X \mid S) = \mathrm{Bin}(S, p)}$$

Proof for the discrete example

Proof: we have seen that for $n \leqslant m$

$$\mathbf{P}(X = n | S = m) = \binom{m}{n} p^n (1-p)^{m-n} \quad \text{with } p = \frac{\lambda}{\lambda + \mu}.$$

Then we consider

- State space $= \mathbb{N}$, $\mathcal{G} = \sigma(S)$

and we verify that these conditional probabilities define a CRL.