Outline

Right inverse (1)

Definition 27.

Let $F : \mathbb{R} \to [0,1]$ continuous cdf
We define the right inverse F^{-1} as

$$F^{-1} : (0,1) \to \mathbb{R}, \quad y \mapsto \inf\{a \in \mathbb{R};\, F(a) \geq y\}$$

Right inverse (2)

Remarks on right inverse:

(i) If F is strictly increasing, F^{-1} is the inverse of F
$\hookrightarrow$ i.e. $F \circ F^{-1} = F^{-1} \circ F = \mathrm{Id}$

(ii) Graphical method to construct F^{-1}:

1. Symmetry wrt diagonal
2. Then erase vertical parts

Example: $F(x) = (x-1)\mathbf{1}_{[1,2)}(x) + \mathbf{1}_{[2,\infty)}(x)$
$\hookrightarrow F^{-1}(y) = (1+y)\mathbf{1}_{(0,1)}(y)$

Right inverse (3)

$\overline{\{x;\, f(x)>0\}}$

More remarks:

(iii) Interpretation:

- In above example, $F \equiv$ cdf of $\mathcal{U}([1,2])$
- Domain of interest: $x \in [1,2]$
- In this domain, we do have $F^{-1}(F(x)) = x$

(iv) Generalization:

If $\mu(dx) = f(x)\, dx$ with $\mathrm{Supp}(f) = [a,b]$, then

- F is strictly increasing on $[a,b]$
- $F : (a,b) \to (0,1)$ is invertible
- One can ignore the set $(a,b)^c$ in order to compute F^{-1}

Inverse method for simulation

Proposition 28.

Let

- μ a continuous probability measure on $\mathbb{R}$
- $F(x) = \mu((-\infty, x])$ with right inverse F^{-1}
- $U \sim \mathcal{U}([0,1])$

Then

$$X = F^{-1}(U) \text{ is distributed according to } \mu$$

Proof Set $X = F^{-1}(U)$. Then

$$\begin{aligned}\mathbb{P}(X \le x) &= \mathbb{P}(F^{-1}(U) \le x) \\ &= \mathbb{P}(U \le F(x)) \\ &= \mathbb{P}(0 \le U \le F(x)) \\ &= \int_0^{F(x)} du \\ &= F(x)\end{aligned}$$

?

Thus $\boxed{X \sim \mu}$

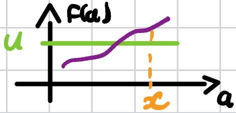
$$\mathbb{P}(F^{-1}(U) \leq x) \overset{?}{\leq} \mathbb{P}(U \leq F(x))$$

One can prove a stronger result:

$$\{u \in [0,1];\ F^{-1}(u) \leq x\} \equiv A_1$$

$$= \{u \in [0,1];\ u \leq F(x)\} \equiv A_2$$

$A_1 \subset A_2$:



$$F^{-1}(u) \leq x \Rightarrow \inf\{a;\ F(a) \geq u\} \leq x$$

$$\Rightarrow \exists\ a_1 \leq x \text{ s.t. } F(a_1) \geq u$$

$$\Rightarrow F(x) \geq F(a_1) \geq u \Rightarrow F(x) \geq u$$

$\{u \in [0,1] ;\ F^{-1}(u) \leq x\} \equiv A_1$

$= \{u \in [0,1] ;\ u \leq F(x)\} \equiv A_2$

$A_2 \subset A_1$:

$$u \leq F(x) \Rightarrow F(x) \geq u$$

$$\Rightarrow \inf\{a ;\ F(a) \geq u\} \leq x$$

$$\Rightarrow F^{-1}(u) \leq x$$

Conclusion: $F^{-1}(u) \leq x \iff u \leq F(x)$

We can also prove $F^{-1}(u) > x \iff u > F(x)$

Proof of Proposition 28 (1)

Strategy: We will prove that

$$\begin{aligned} \mathbf{P}(X \leq x) &= \mathbf{P}(F^{-1}(U) \leq x) \\ &\overset{(*)}{=} \mathbf{P}(U \leq F(x)) = F(x) \end{aligned}$$

Details for $(*)$

We wish to show that for $x \in \mathbb{R}$,

$$\left\{u \in (0,1);\ F^{-1}(u) \leq x\right\} = \{u \in (0,1);\ u \leq F(x)\}$$

Inclusion $\subset$:

$$\begin{aligned} F^{-1}(u) \leq x \quad &\Rightarrow \quad \inf\{a;\ F(a) \geq u\} \leq x \\ &\Rightarrow \quad \text{There exists } a_1 \leq x \text{ such that } F(a_1) \geq u \\ &\Rightarrow \quad F(x) \geq F(a_1) \geq u \end{aligned}$$

Inclusion $\supset$:

$$\begin{aligned} u \leq F(x) \quad &\Rightarrow \quad F(x) \geq u \\ &\Rightarrow \quad \inf\{a;\ F(a) \geq u\} \leq x \\ &\Rightarrow \quad F^{-1}(u) \leq x \end{aligned}$$

Example: $\mu = \mathcal{U}([a,b])$

$$f(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$$

$\Rightarrow$ On $[a,b]$, $F(x) = \dfrac{x-a}{b-a} = \int_a^x f(r)\,dr$

$\Rightarrow$ On $[a,b]$

$$F(x) = y \Leftrightarrow \frac{x-a}{b-a} = y$$

$$\Leftrightarrow x = a + (b-a)y = F^{-1}(y)$$

$\Rightarrow$ If $U \sim \mathcal{U}([0,1])$ and $X = a + (b-a)U \Rightarrow X \sim \mathcal{U}([a,b])$

Example 2

Exponential (λ), density

$$f(x) = \lambda e^{-\lambda x} \mathbb{1}_{[0,\infty)}(x)$$

If $x \in [0,\infty)$,

$$F(x) = \int_0^x \lambda e^{-\lambda r} dr = 1 - e^{-\lambda x}$$

$$F(x) = y \Leftrightarrow 1 - e^{-\lambda x} = y$$

$$\Leftrightarrow e^{-\lambda x} = 1 - y \Leftrightarrow x = -\frac{1}{\lambda} \ln(1-y)$$

$\Rightarrow$ If $U \sim \mathcal{U}([0,1])$, and $X = -\frac{1}{\lambda} \ln(1-U)$ we have $X \sim \mathcal{E}(\lambda)$

Examples

Example 1:
Let $\mu = \mathcal{U}([a, b])$. Then on $[a, b]$

$$F(x) = \frac{x-a}{b-a}, \quad \text{and} \quad F^{-1}(y) = a + (b-a)y$$

One can check that $X = a + (b-a)U \sim \mathcal{U}([a, b])$

Example 2:
Let $\mu = \mathcal{E}(\lambda)$. Then on $\mathbb{R}_+$

$$F(x) = \int_0^x \lambda e^{-\lambda u}\, du = 1 - e^{-\lambda x}, \quad \text{and} \quad F^{-1}(y) = -\frac{\ln(1-y)}{\lambda}$$

One can check that $X = -\frac{\ln(1-U)}{\lambda} \sim \mathcal{E}(\lambda)$

Comments on inverse method

Pros:

- Unique call to `rand`
- Excellent simulation method ... when it works!

Cons:

- Explicit computation of F, F^{-1} not always possible
- Typical example: $\mathcal{N}(0,1)$

Examples of application:

Exponential, Weibull, Cauchy

Outline

Skorohod's representation theorem

Proposition 29.

Consider

- $\{X_n;\, n \geq 1\}$ sequence such that $X_n \xrightarrow{(d)} X$

Then one can construct

1. A probability space $(\Omega, \mathcal{F}, \mathbf{P})$
2. Random variables $Y_n : \Omega \to \mathbb{R}$ satisfying $Y_n \overset{(d)}{=} X_n$
3. $Y : \Omega \to \mathbb{R}$ satisfying $Y \overset{(d)}{=} X$

such that the following holds true:

$$Y_n \xrightarrow{a.s} Y$$

Lebesgue measure

Proof We take

$\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, $\mathbb{P} = \lambda$ (↑ Lebesgue measure)

↳ canonical space for $\mathcal{U}([0,1])$

Recall: Each X_n has a cdf F_n
X has a cdf F

Define ($\omega \in [0,1]$)

$$Y_n(\omega) = F_n^{-1}(\omega) \sim X_n$$

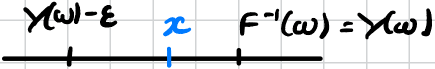
$$Y(\omega) = F^{-1}(\omega) \sim X$$

Claim: If ω is a point of continuity of F^{-1}, then

$$\lim_{n\to\infty} Y_n(\omega) = \lim_{n\to\infty} F_n^{-1}(\omega)$$

$$= F^{-1}(\omega) = Y(\omega)$$

1st step of proof. Take $\omega \in [0,1]$



→ x s.t. x point of continuity of F

$$F^{-1}(\omega) > x \Rightarrow \omega > F(x) \Rightarrow F(x) < \omega$$

$$\Rightarrow F_n(x) < \omega \text{ for } n \text{ large} \Rightarrow x < F_n^{-1}(\omega)$$

Summary For n large enough

$$Y(\omega)-\varepsilon < x < F_n^{-1}(\omega) = Y_n(\omega)$$

$$\Rightarrow \quad \liminf_n Y_n(\omega) \geq Y(\omega) - \varepsilon \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad \boxed{\liminf Y_n(\omega) \geq Y(\omega)}$$

F is $\nearrow$ $\Rightarrow$ F has at most a countable # of points of discontinuity

Summary 2 We have proved

$$\liminf Y_n(\omega) \geq Y(\omega)$$

One can also prove that if $\omega' > \omega$

$$\limsup Y_n(\omega) \leq Y(\omega')$$

Thus, for all $\omega' > \omega$

$$Y(\omega) \leq \liminf Y_n(\omega) \leq \limsup Y_n(\omega) \leq Y(\omega')$$

If ω is a point of cont. for F^{-1}, we get

$$\boxed{\lim Y_n(\omega) = Y(\omega)}$$

If ω is a point of cont. for F^{-1}, veget
$\lim Y_n(\omega) = Y(\omega)$

Define $D = \{ \omega \in \Omega ; F^{-1}$ is discontinuous at $\omega \}$

F^{-1} is non $\searrow$ $\Rightarrow$ D at most countable

$\Rightarrow \lambda(D) = 0 \Rightarrow \mathbb{P}(D) = 0$

$\Rightarrow$ a.s ω is a point of continuity for F^{-1}

$\Rightarrow$ $Y_n \longrightarrow Y$ a.s.

Proof of Proposition 29 (1)

Definition of $(\Omega, \mathcal{F}, \mathbf{P})$: We take

$$\Omega = [0,1], \quad \mathcal{F} = \text{ Borel } \sigma\text{-algebra}, \quad \mathbf{P} = \lambda$$

Definition of Y_n and Y: We take

$$Y_n(\omega) = F_n^{-1}(\omega), \qquad Y(\omega) = F^{-1}(\omega)$$

Distributions of Y_n and Y: According to Proposition 28,

$$Y_n \sim F_n, \qquad Y \sim F$$

Proof of Proposition 29 (2)

Claim 1: If ω is a point of continuity of F^{-1}, we have

$$\lim_{n\to\infty} Y_n(\omega) = \lim_{n\to\infty} F_n^{-1}(\omega) = F^{-1}(\omega) = Y(\omega) \tag{1}$$

Proof of claim 1: Consider

- $\omega \in [0,1]$
- x point of continuity of F such that $Y(\omega) - \varepsilon < x < Y(\omega)$

We have

$$\begin{aligned} F^{-1}(\omega) > x &\implies F(x) < \omega \\ &\implies F_n(x) < \omega, \quad \text{for large } n \\ &\implies x < F_n^{-1}(\omega), \quad \text{for large } n \end{aligned}$$

Proof of Proposition 29 (3)

Proof of claim 1 - ctd: We have seen, for n large enough,

$$Y(\omega) - \varepsilon < x < F_n^{-1}(\omega) \quad \left(\Longrightarrow \quad F_n^{-1}(\omega) > Y(\omega) - \varepsilon\right)$$

Partial conclusion: We get

$$\liminf_{n\to\infty} Y_n(\omega) > Y(\omega) - \varepsilon, \quad \text{for all } \varepsilon > 0$$
$$\Longrightarrow \liminf_{n\to\infty} Y_n(\omega) \geq Y(\omega)$$

Proof of Proposition 29 (4)

Proof of claim 1 - ctd: We have proved

$$\liminf_{n\to\infty} Y_n(\omega) \geq Y(\omega)$$

Along the same lines, for $\omega' > \omega$ one has

$$\limsup_{n\to\infty} Y_n(\omega) \leq Y(\omega')$$

Conclusion: Claim 1 is true, that is
$\hookrightarrow$ If ω is a point of continuity of F^{-1}, we have

$$\lim_{n\to\infty} Y_n(\omega) = Y(\omega) \tag{2}$$

Proof of Proposition 29 (5)

Almost sure convergence: Let

$$D = \left\{\text{points of discontinuity of } F^{-1}\right\}$$

Since F^{-1} non decreasing,

$$\mathbf{P}(D) = \lambda(D) = 0$$

Hence

$$Y_n \xrightarrow{\text{a.s}} Y$$

Characterization of convergence in distribution

Proposition 30.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables

Then the statements 1-2-3 are equivalent:

1. $X_n \xrightarrow{(d)} X$
2. For any $f \in C_b(\mathbb{R})$, we have continuous + bounded (weak convergence)
$$\lim_{n\to\infty} \mathbf{E}\left[f(X_n)\right] = \mathbf{E}\left[f(X)\right]$$
3. For every $u \in \mathbb{R}$ we have
$$\lim_{n\to\infty} \mathbf{E}\left[e^{\imath u X_n}\right] = \mathbf{E}\left[e^{\imath u X}\right]$$

Proof of $F_n \to F \implies E[f(X_n)] \to E[f(X)]$ $\forall f \in C_b(\mathbb{R})$

By Skorohod, let $Y_n \sim X_n$, $Y \sim X$ s.t.

$$Y_n \xrightarrow{a.s.} Y$$

Since f continuous, we have

$$f(Y_n) \xrightarrow{a.s.} f(Y)$$

Since f is bounded, by bounded convergence

$$\boxed{\lim E[f(Y_n)] = E[f(Y)]}$$