Characterization of convergence in distribution

Proposition 30.

Consider

- $\{X_n;\, n \geq 1\}$ sequence of random variables

Then the statements 1-2-3 are equivalent:

1. $X_n \xrightarrow{(d)} X$
2. For any $f \in C_b(\mathbb{R})$, we have
$$\lim_{n\to\infty} \mathbf{E}\left[f(X_n)\right] = \mathbf{E}\left[f(X)\right]$$
3. $u \mapsto \mathbf{E}\left[e^{\imath u X}\right]$ cont. at 0, and for every $u \in \mathbb{R}$ we have
$$\lim_{n\to\infty} \mathbf{E}\left[e^{\imath u X_n}\right] = \mathbf{E}\left[e^{\imath u X}\right]$$

Proof of Proposition 30 (1)

Application of Skorohod: One can find

$$Y_n \stackrel{(d)}{=} X_n, \qquad Y \stackrel{(d)}{=} X$$

such that

$$Y_n \xrightarrow{a.s} Y$$

Convergence of $g(Y_n)$: Since g is continuous, we have

$$g(Y_n) \xrightarrow{a.s} g(Y)$$

Proof of $1 \Longrightarrow 2$: By bounded convergence,

$$\mathbf{E}[g(Y_n)] \longrightarrow \mathbf{E}[g(Y)]$$

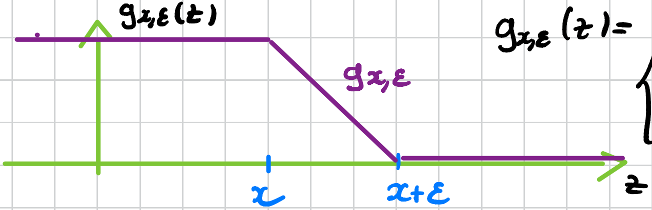
Aim: Assume $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ for all $f \in C_b$. Prove that

$$F_n(x) \to F(x) \quad \forall x \text{ point of cont.}$$

Problem

- $F_n(x) = \mathbb{E}[\mathbb{1}_{(-\infty,x]}(X_n)]$
- However, $z \mapsto \mathbb{1}_{(-\infty,x]}(z)$ is not continuous

Strategy: approximate $\mathbb{1}_{(-\infty,x]}$ by C_b-functions



$$g_{x,\varepsilon}(z) = \begin{cases} 1 & \text{if } z \le x \\ 0 & \text{if } z > x+\varepsilon \\ \text{linear between} & x \text{ and } x+\varepsilon \end{cases}$$

We have, for all $\varepsilon > 0$

$$\mathbb{1}_{[-\infty, x]}(X_n) \le g_{x,\varepsilon}(X_n)$$

$$\Rightarrow F_n(x) = \mathbb{E}[\mathbb{1}_{(-\infty,x]}(X_n)] \le \mathbb{E}[g_{x,\varepsilon}(X_n)]$$

$$\Rightarrow \boxed{\limsup_n F_n(x) \le} \limsup_n \mathbb{E}[g_{x,\varepsilon}(X_n)]$$

$$= \mathbb{E}[g_{x,\varepsilon}(X)]$$

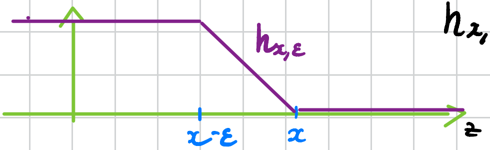
$$\le \mathbb{E}[\mathbb{1}_{(-\infty, x+\varepsilon]}(X)] = \boxed{F(x+\varepsilon)}$$

We have obtained

$$\limsup_n F_n(x) \leq F(x+\varepsilon) \quad \forall \varepsilon > 0$$

$$\Rightarrow \quad \boxed{\limsup_n F_n(x) \leq F(x)}$$

Study of liminf



$$h_{x,\varepsilon} \leq \mathbb{1}_{(-\infty, x]}$$

$$\mathbb{1}_{(-\infty,x]}(X_n) \geq h_{x,\varepsilon}(X_n)$$

$$\overset{\mathbb{E}}{\Longrightarrow} \quad F_n(x) \geq \mathbb{E}[h_{x,\varepsilon}(X_n)]$$

$$\overset{\text{liminf}}{\Longrightarrow} \quad \liminf F_n(x) \geq \mathbb{E}[h_{x,\varepsilon}(X)]$$

$$\geq F(x-\varepsilon) \quad \forall \varepsilon > 0$$

If x point of cont. of F, we get

$$\boxed{\liminf F_n(x) \geq F(x)}$$

Summary We have obtained, if x is a point of cont. of F,

$$F(x) \leq \liminf_n F_n(x) \leq \limsup_n F_n(x) \leq F(x)$$

$$\Rightarrow \lim_{n\to\infty} F_n(x) = F(x)$$

$$\Rightarrow \boxed{X_n \xrightarrow{(d)} X}$$

Proof of Proposition 30 (2)

Next step:

Proof of $1 \Longrightarrow 2$

Approximation of $\mathbf{1}_{(-\infty,x]}$: For $\varepsilon > 0$ we set

$$g_{\varepsilon,x}(y) = \begin{cases} 1, & \text{if } y \leq x \\ 0, & \text{if } y \geq x+\varepsilon \\ \text{linear}, & \text{if } x \leq y \leq x+\varepsilon \end{cases}$$

Upper bound for F_n: For $x \in \mathbb{R}$ we have

$$g_{x,\varepsilon}(y) \geq \mathbf{1}_{(y\leq x)}$$
$$\Longrightarrow F_n(x) = \mathbf{E}\left[\mathbf{1}_{(X_n\leq x)}\right] \leq \mathbf{E}\left[g_{\varepsilon,x}(X_n)\right]$$

Proof of Proposition 30 (3)

Taking $\limsup$: Since we assume 2 holds,

$$\begin{aligned}\limsup_{n\to\infty} F_n(x) &\le \limsup_{n\to\infty} \mathbf{E}\left[g_{x,\varepsilon}(X_n)\right] \\ &\le \mathbf{P}\left(X \le x+\varepsilon\right) \\ &= F(x+\varepsilon)\end{aligned}$$

Taking limits in ε: For all x we end up with

$$\limsup_{n\to\infty} F_n(x) \le F(x)$$

Proof of Proposition 30 (4)

Taking lim inf: By considering $g_{x-\varepsilon,\varepsilon}$ we obtain

$$\begin{aligned}
\liminf_{n\to\infty} F_n(x) &\geq \liminf_{n\to\infty} \mathbf{E}\left[g_{x-\varepsilon,\varepsilon}(X_n)\right] \\
&\geq \mathbf{P}\left(X \leq x-\varepsilon\right) \\
&= F(x-\varepsilon)
\end{aligned}$$

Taking limits in ε: For a continuity point x of F, we get

$$\liminf_{n\to\infty} F_n(x) \geq F(x)$$

Conclusion: For a continuity point x of F, we have

$$\lim_{n\to\infty} F_n(x) = F(x)$$