Review session - MA 538 - Midterm

Problem 6. Let $X = \{X_n; n \in \mathbb{N}\}$ be a stochastic process such that for $k \geq 2$ and $0 = n_0 < n_1 < \cdots < n_k$, the random variables $(\delta X_{n_j n_{j+1}})_{0 \leq j \leq n-1}$ are independent (here we have set $\delta X_{n_j n_{j+1}} = X_{n_{j+1}} - X_{n_j}$). We also assume that $X_0 = 0$. Show that for all $0 \leq m < n < \infty$, the random variable δX_{mn} is in fact independent of the whole σ-field $\mathcal{F}_m^X = \sigma(X_1, \ldots, X_m)$.

Known information

$$\mathbb{P}(\delta X_{01} \in A_1, \ldots, \delta X_{m-1,m} \in A_m, \delta X_{mn} \in A_n)$$

$$= \prod_{j=0}^{m-1} \mathbb{P}(\delta X_{j,j+1} \in A_{j+1}) \times \mathbb{P}(\delta X_{mn} \in A_n)$$

Definition of a π-system

We set

$$\mathcal{P} = \{ B \equiv (\delta X_{01} \in A_1, \ldots, \delta X_{m-1,m} \in A_m);$$
$$A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}) \}$$

One can prove easily that

$$B, \hat{B} \in \mathcal{P} \implies B \cap \hat{B} \in \mathcal{P}$$

Definition of a λ-system For $C \in \mathcal{B}(\mathbb{R})$, set

$$\mathcal{L} = \{ B ;\ \mathbb{P}(B \cap (\delta X_{mn} \in C)) = \mathbb{P}(B)\, \mathbb{P}(\delta X_{mn} \in C) \}$$

Then $\mathcal{B}$ is a λ-system

Application of Dykin

Since $\mathcal{L} \supset \mathcal{P}$, we also have

$$\mathcal{L} \supset \sigma(\mathcal{P})$$

$$\Rightarrow \mathcal{L} \supset \sigma(\delta X_{01}, \dots, \delta X_{m-1,m})$$

Thus $\boxed{\delta X_{mn} \perp\!\!\!\perp \sigma(\delta X_{01}, \dots, \delta X_{m-1,m})}$

Statement with $\sigma(X_1, \dots, X_m)$ we still need to prove

$$\sigma(\delta X_{01}, \dots, \delta X_{m-1,m}) = \sigma(X_1, \dots, X_m)$$

In fact we will prove, for a r.v. Y

$$\boxed{Y \in \sigma(\delta X_{01}, \dots, \delta X_{m-1,m}) \Leftrightarrow Y \in \sigma(X_1, \dots, X_m)}$$

If $Y \in \sigma(\delta X_{01}, .., \delta X_{m-1,m})$ There exists a measurable ψ such that

$$Y = \psi(\delta X_{01}, \dots, \delta X_{m-1,m})$$

Thus

$$Y = \psi(X_1, X_2 - X_1, .., X_m - X_{m-1})$$

This is of the form

$$Y = \varphi(X_1, .., X_m),$$

for a measurable φ. Hence

$$\boxed{Y \in \sigma(X_1, \dots, X_m)}$$

If $Y \in \sigma(X_1, .., X_m)$ In the same way, we have, for a measurable f,

$$Y = f(X_1, \dots, X_m)$$

Then

$$Y = f\left(\delta X_{01}, \delta X_{01} + \delta X_{12}, .., \sum_{j=0}^{m-1} \delta X_{j,j+1}\right)$$

$= g(\delta X_{01}, .., \delta X_{m-1,m})$, with g measurable

$$\Rightarrow \boxed{Y \in \sigma(\delta X_{01}, .., \delta X_{m-1,m})}$$

Problem 14. Give a rigorous proof that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ for any pair X, Y of independent non-negative random variables in $L^1(\Omega)$.
Hint: For $k \geq 0, n \geq 1$, define $X_n = k/n$ if $k/n \leq X < (k+1)/n$, and similarly for Y_n. Show that X_n and Y_n are independent, and $X_n \leq X$, and $Y_n \leq Y$. Deduce that $\mathbf{E}[X_n] \to \mathbf{E}[X]$ and $\mathbf{E}[Y_n] \to \mathbf{E}[Y]$, and also $\mathbf{E}[X_n Y_n] \to \mathbf{E}[XY]$.

Computation for X_n, Y_n X_n, Y_n are discrete r.v. Hence

$$\mathbb{E}[X_n Y_n] = \sum_{j,k=0}^{\infty} \frac{j}{n} \frac{k}{n} \mathbb{P}(X \in I_j^n, Y \in I_k^n),$$

where we have set

$$I_j^n = \left[\frac{j}{n}, \frac{j+1}{n}\right].$$

Furthermore

$$\mathbb{P}(X \in I_j^n, Y \in I_k^n) \overset{X \perp Y}{=} \mathbb{P}(X \in I_j^n)\, \mathbb{P}(Y \in I_k^n)$$

Hence

$$\boxed{\mathbb{E}[X_n Y_n]} = \sum_{j,k=0}^{\infty} \frac{j}{n} \frac{k}{n} \mathbb{P}(X \in I_j^n)\, \mathbb{P}(Y \in I_k^n)$$

$$= \sum_{j=0}^{\infty} \frac{j}{n} \mathbb{P}(X \in I_j^n) \times \sum_{k=0}^{\infty} \frac{k}{n} \mathbb{P}(Y \in I_k^n)$$

$$\boxed{= \mathbb{E}[X_n]\, \mathbb{E}[Y_n]}$$

Note we have $X_n \perp\!\!\!\perp Y_n$. Indeed,

$$X_n = \sum_{j=0}^{\infty} \frac{j}{n} \mathbb{1}_{[\frac{j}{n}, \frac{j+1}{n}]}(X) = \varphi_n(X),$$

where φ_n is measurable. We also have $Y_n = \varphi_n(Y)$. Hence

$$X \perp\!\!\!\perp Y \implies \varphi_n(X) \perp\!\!\!\perp \varphi_n(Y)$$

$$\implies \boxed{X_n \perp\!\!\!\perp Y_n}$$

Taking limits we have seen that

$$\mathbb{E}[X_n Y_n] = \mathbb{E}[X_n]\,\mathbb{E}[Y_n]$$

One can take monotone convergence on both sides and get

$$\boxed{\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]}$$

33.2. Let $\{X_n\}$ be a stationary Markov chain on the positive integers with transition probabilities

$$p_{jk} = \begin{cases} \frac{j}{j+2} & \text{if } k = j+1 \\ \frac{2}{j+2} & \text{if } k = 1 \end{cases}$$

(1) Find the stationary distribution of the chain, and show that it has infinite mean.
(2) Show that $\limsup_{r\to\infty} X_r/r \leq 1$ almost surely.

Stationary distribution of the form $\{\pi_k;\ k \geq 1\}$ such that

$$\pi P = \pi$$

$$\Leftrightarrow \sum_{j=1}^{\infty} \pi_j \, p_{jk} = \pi_k \qquad \forall j \geq 1$$

$$\Leftrightarrow \begin{cases} \pi_{k-1} \dfrac{k-1}{k+1} = \pi_k & \text{for } j \geq 2 \\ \displaystyle\sum_{j=1}^{\infty} \pi_j \frac{2}{j+1} = \pi_1 \end{cases}$$

Solving the system we get

$$\pi_k = \frac{k-1}{k+1}\pi_{k-1} = \frac{k-1}{k+1}\,\frac{k-2}{k}\,\pi_{k-2}$$

$$= \frac{(k-1)(k-2)}{(k+1)k}\,\frac{(k-3)}{(k-1)}\,\pi_{k-3}$$

$$= \frac{\prod_{j=1}^{k-1} j}{\prod_{j=3}^{k+1} j}\,\pi_1 = \frac{2}{(k+1)k}\,\pi_1$$

$$\Rightarrow \quad \pi_k = \frac{2}{k(k+1)}\,\pi_1$$