MA 538 - PROBLEM LIST

PROBLEMABILITY THEORY 1

1. Basic notions of probability

Problem 1. Let $\gamma_{a,b}$ be the function:

$$\gamma_{a,b}(x) = \frac{1}{\Gamma(a) b^a} x^{a-1} e^{-x/b} 1_{\{x>0\}},$$

where $a, b > 0$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

1.1. Show that $\gamma_{a,b}$ is a density.

1.2. Let X a random variable with density $\gamma_{a,b}$. Check, for $\lambda > 0$:

$$\mathbf{E}[e^{-\lambda X}] = \frac{1}{(1+\lambda b)^a}, \qquad \mathbf{E}[X] = ab, \qquad VarX = ab^2.$$

1.3. Let X (resp. X') a random variable with density $\gamma_{a,b}$ (resp. $\gamma_{a',b}$). We assume X and X' independent. Show that $X + X'$ admits the density $\gamma_{a+a',b}$.

1.4. Application: Let $X_1, X_2, \ldots, X_n, n$ i.i.d random variables, with law $\mathcal{N}(0,1)$. Show that $X_1^2 + X_2^2 + \cdots + X_n^2$ is Gamma distributed.

Problem 2. Let X be a random variable distributed as $\mathcal{N}_1(m, \sigma^2)$.

2.1. Assume $m = 0$. We set $Y = e^{\alpha X^2}$ with $\alpha \neq 0$. Compute $E[Y^n]$.

2.2. Find the density of $|X - 1|$ when $m = 2$ and $\sigma = 1$.

Problem 3. Let X be a random variable which takes non-negative values only. Show that

$$\sum_{i=1}^\infty (i-1)\mathbf{1}_{A_i} \leq X < \sum_{i=1}^\infty i\mathbf{1}_{A_i} \,,$$

where $A_i = \{i - 1 \leq X < i\}$. Deduce that

$$\sum_{i=1}^\infty \mathbf{P}(X \geq i) \leq \mathbf{E}(X) < 1 + \sum_{i=1}^\infty \mathbf{P}(X \geq i)$$

Problem 4. One can often find the definition of λ-system under the following form: we declare that $\mathcal{L}$ is a λ-system if:

(1) $\Omega \in \mathcal{L}$.
(2) If $A, B \in \mathcal{L}$ and $B \subset A$, then $A \setminus B \in \mathcal{L}$.
(3) If $(A_n)_{n\geq 1}$ is an increasing sequence of elements of $\mathcal{L}$, then $\cup_{n\geq 1} A_n \in \mathcal{L}$.

Show that this definition is equivalent to the one seen in class.

Problem 5. We consider a measurable space $(\Omega, \mathcal{F})$ and some families $\{\mathcal{A}_i; i \le n\}$ of subsets of Ω such that $\mathcal{A}_i \subset \mathcal{F}$.

5.1. Show that if the $\mathcal{A}_i$ are independent and each one is a π-system, then the σ-algebras $\sigma(\mathcal{A}_i)$ are independent.

5.2. Let $\{X_i; i \le n\}$ be a collection of real valued random variables. Show that these random variables are independent if and only if

$$\mathbf{P}\left(X_1 \le x_1, \dots, X_n \le x_n\right) = \prod_{i \le n} \mathbf{P}\left(X_i \le x_i\right)$$

for any vector $(x_1, \dots, x_n) \in \mathbb{R}^n$.

Problem 6. Let $X = \{X_n; n \in \mathbb{N}\}$ be a stochastic process such that for $k \ge 2$ and $0 = n_0 < n_1 < \cdots < n_k$, the random variables $(\delta X_{n_j n_{j+1}})_{0 \le j \le n-1}$ are independent (here we have set $\delta X_{n_j n_{j+1}} = X_{n_{j+1}} - X_{n_j}$). We also assume that $X_0 = 0$. Show that for all $0 \le m < n < \infty$, the random variable δX_{mn} is in fact independent of the whole σ-field $\mathcal{F}_m^X = \sigma(X_1, \dots, X_m)$.

2. Modes of convergence

Problem 7. Let $r \ge 1$, and define $\|X\|_r = \{\mathbf{E}\,|X^r|\}^{1/r}$. Show that

7.1. $\|cX\|_r = |c| \cdot \|X\|_r$ for $c \in \mathbb{R}$.

7.2. $\|X + Y\|_r \le \|X\|_r + \|Y\|_r$.

7.3. $\|X\|_r = 0$ if and only if $\mathbf{P}(X = 0) = 1$.

This amounts to saying that $\|\cdot\|_r$ is a norm on the set of equivalence classes of random variables on a given probability space with finite r th moment, the equivalence relation being given by $X \sim Y$ if and only if $\mathbf{P}(X = Y) = 1$.

Problem 8. Define $\langle X, Y\rangle = \mathbf{E}[X\,Y]$ for random variables X and Y having finite variance, and define $\|X\| = \langle X, X\rangle^{1/2}$. Show that

8.1. $\langle aX + bY, Z\rangle = a\langle X, Z\rangle + b\langle Y, Z\rangle$.

8.2. $\|X + Y\|^2 + \|X - Y\|^2 = 2\left(\|X\|^2 + \|Y\|^2\right)$, the parallelogram property.

8.3. If $\langle X_i, X_j\rangle = 0$ for all $i \ne j$ then

$$\left\|\sum_{i=1}^n X_i\right\|^2 = \sum_{i=1}^n \|X_i\|^2 .$$

These properties yield a *Hilbert space structure* on $L^2(\Omega)$.

Problem 9. Let $\epsilon > 0$. Let $g, h : [0,1] \to \mathbb{R}$, and define $d_\epsilon(g,h) = \int_E dx$ where $E = \{u \in [0,1] : |g(u) - h(u)| > \epsilon\}$. Show that d_ϵ does not satisfy the triangle inequality.

Problem 10. For two distribution functions F and G, let

$$d(F,G) = \inf\{\delta > 0 : F(x-\delta) - \delta \le G(x) \le F(x+\delta) + \delta \text{ for all } x \in \mathbb{R}\}.$$

Show that d is a metric on the space of distribution functions. It is named *Lévy metric.*

Problem 11. Find random variables $X, X_1, X_2, \ldots$ such that $\mathbf{E}\left[|X_n - X|^2\right] \to 0$ as $n \to \infty$, but $\mathbf{E}\left[|X_n|\right] = \infty$ for all n.

Problem 12. We consider a sequence $\{X_n;\, n \ge 1\}$ of random variables.

12.1. Suppose $X_n \xrightarrow{r} X$ where $r \ge 1$. Show that $\mathbf{E}[|X_n^r|] \to \mathbf{E}\left[|X^r|\right]$.

12.2. Suppose $X_n \xrightarrow{1} X$. Show that $\mathbf{E}\left[X_n\right] \to \mathbf{E}[X]$. Is the converse true?

12.3. Suppose $X_n \xrightarrow{2} X$. Show that $\operatorname{var}(X_n) \to \operatorname{var}(X)$.

Problem 13. Suppose $|X_n| \le Z$ for all n, where $\mathbf{E}(Z) < \infty$. Prove that if $X_n \xrightarrow{\mathrm{P}} X$ then $X_n \xrightarrow{1} X$. This is called the *Dominated Convergence Theorem.*

Problem 14. Give a rigorous proof that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ for any pair X, Y of independent non-negative random variables in $L^1(\Omega)$.
Hint: For $k \ge 0, n \ge 1$, define $X_n = k/n$ if $k/n \le X < (k+1)/n$, and similarly for Y_n. Show that X_n and Y_n are independent, and $X_n \le X$, and $Y_n \le Y$. Deduce that $\mathbf{E}[X_n] \to \mathbf{E}[X]$ and $\mathbf{E}[Y_n] \to \mathbf{E}[Y]$, and also $\mathbf{E}\left[X_n Y_n\right] \to \mathbf{E}[XY]$.

Problem 15. Show that convergence in distribution is equivalent to convergence with respect to the Lévy metric of Problem 10.

Problem 16. We consider a sequence $\{X_n;\, n \ge 1\}$ of random variables.

16.1. Suppose that $X_n \xrightarrow{(\mathrm{d})} X$ and $Y_n \xrightarrow{\mathrm{P}} c$, where c is a constant. Show that $X_n Y_n \xrightarrow{(\mathrm{d})} cX$, and that $X_n / Y_n \xrightarrow{(\mathrm{d})} X/c$ if $c \neq 0$.

16.2. Suppose that $X_n \xrightarrow{(\mathrm{d})} 0$ and $Y_n \xrightarrow{\mathrm{P}} Y$, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $y \mapsto g(x,y)$ is a continuous function of y for all x, and $(x,y) \mapsto g(x,y)$ is continuous at every point of the form $(0,y)$. Show that $g\left(X_n, Y_n\right) \xrightarrow{\mathrm{P}} g(0,Y)$.

These results are sometimes referred to as *Slutsky's theorem(s).*

Problem 17. Let $\{X_n;\, n \ge 1\}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Show that the set $A = \{\omega \in \Omega : \text{the sequence } X_n(\omega) \text{ converges }\}$ is an event (that is, lies in $\mathcal{F}$), and that there exists a random variable X (that is, an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$) such that $X_n(\omega) \to X(\omega)$ for $\omega \in A$.

Problem 18. Let $\{X_n;\, n \geq 1\}$ be a sequence of random variables, and let $\{c_n;\, n \geq 1\}$ be a sequence of reals converging to the limit c. For convergence almost surely, in r th mean, in probability, and in distribution, show that the convergence of X_n to X entails the convergence of $c_n X_n$ to cX.

Problem 19. Let $\{X_n;\, n \geq 1\}$ be a sequence of independent random variables which converges in probability to the limit X. Show that X is almost surely constant.

Problem 20. The sequence of discrete random variables X_n, with mass functions f_n, is said to *converge in total variation* to X with mass function f if

$$\sum_x |f_n(x) - f(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Suppose $X_n \to X$ in total variation, and $u : \mathbb{R} \to \mathbb{R}$ is bounded. Show that

$$\mathbf{E}\left[u\left(X_n\right)\right] \to \mathbf{E}[u(X)].$$

Problem 21. Let $\{X_n;\, n \geq 1\}$ be independent Poisson variables with respective parameters $\{\lambda_n;\, n \geq 1\}$. Show that $\sum_{n=1}^{\infty} X_n$ converges or diverges almost surely according as $\sum_{n=1}^{\infty} \lambda_n$ converges or diverges.

3. Laws of large numbers

Problem 22. We consider a sequence $\{X_n;\, n \geq 1\}$ of random variables.

22.1. Suppose that $X_n \xrightarrow{\mathrm{P}} X$. Show that $\{X_n\}$ is Cauchy convergent in probability. Namely we have that for all $\epsilon > 0, \mathbf{P}\left(|X_n - X_m| > \epsilon\right) \to 0$ as $n, m \to \infty$. In what sense is the converse true?

22.2. Let $\{X_n\}$ and $\{Y_n\}$ be sequences of random variables such that the pairs (X_i, X_j) and (Y_i, Y_j) have the same distributions for all i, j. If $X_n \xrightarrow{\mathrm{P}} X$, show that Y_n converges in probability to some limit Y having the same distribution as X.

Problem 23. Show that the probability that infinitely many of the events $\{A_n : n \geq 1\}$ occur satisfies $\mathbf{P}(A_n \text{ occurs i.o}) \geq \limsup_{n\to\infty} \mathbf{P}\left(A_n\right)$.

Problem 24. Let $\{S_n : n \geq 0\}$ be a simple random walk which moves to the right with probability p at each step, and suppose that $S_0 = 0$. Write $X_n = S_n - S_{n-1}$.

24.1. Show that $\{S_n = 0 \text{ i.o}\}$ is not a tail event of the sequence $\{X_n\}$.

24.2. Show that $\mathbf{P}(S_n = 0 \text{ i.o}) = 0$ if $p \neq \frac{1}{2}$.

24.3. Let $T_n = S_n/\sqrt{n}$, and show that

$$\left\{\liminf_{n\to\infty} T_n \leq -x\right\} \cap \left\{\limsup_{n\to\infty} T_n \geq x\right\}$$

is a tail event of the sequence $\{X_n\}$, for all $x > 0$. Deduce directly that $\mathbf{P}(S_n = 0 \text{ i.o}) = 1$ if $p = \frac{1}{2}$.

Problem 25. Let $\{X_n;\, n \geq 1\}$ be independent identically distributed random variables. The event A, defined in terms of the X_n, is called exchangeable if A is invariant under finite permutations of the coordinates, which is to say that its indicator function $\mathbf{1}_A$ satisfies $\mathbf{1}_A(X_1, X_2, \ldots, X_n, \ldots) = \mathbf{1}_A(X_{i_1}, X_{i_2}, \ldots, X_{i_n}, X_{n+1}, \ldots)$ for all $n \geq 1$ and all permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$. Show that all exchangeable events A are such that either $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$. This result is called *Hewitt-Savage zero-one law*.

Problem 26. Returning to the simple random walk S of Problem 24, show that $\{S_n = 0 \text{ i.o.}\}$ is an exchangeable event with respect to the steps of the walk. Deduce from the Hewitt-Savage zero-one law that it has probability either 0 or 1 .

Problem 27. Let $f : [0,1] \to \mathbb{R}$ be a continuous function, and let S_n be a random variable having the binomial distribution with parameters n and x. Using the formula

$$\mathbf{E}[Z] = \mathbf{E}[Z\,\mathbf{1}_A] + \mathbf{E}[Z\,\mathbf{1}_{A^c}]\ ,$$

with $Z = f(x) - f(n^{-1}S_n)$ and $A = \{|n^{-1}S_n - x| > \delta\}$, show that

$$\lim_{n\to\infty} \sup_{0\leq x\leq 1} \left| f(x) - \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \right| = 0.$$

You have proved *Weierstrass's approximation theorem*, which states that every continuous function on $[0,1]$ may be approximated by a polynomial uniformly over the interval.

Problem 28. A sequence $\{X_n;\, n \geq 1\}$ of random variables is said to be *completely convergent* to X if

$$\sum_n \mathbf{P}\left(|X_n - X| > \epsilon\right) < \infty, \qquad \text{for all } \epsilon > 0.$$

Show that, for sequences of independent variables, complete convergence is equivalent to a.s. convergence. Find a sequence of (dependent) random variables which converges a.s. but not completely.

Problem 29. Let $\{X_n;\, n \geq 1\}$ be independent identically distributed random variables with common mean μ and finite variance. Show that

$$\binom{n}{2}^{-1} \sum_{1\leq i<j\leq n} X_i X_j \xrightarrow{\mathrm{P}} \mu^2, \qquad \text{as } n \to \infty.$$

Problem 30. Let $\{X_n;\, n \geq 1\}$ be independent and exponentially distributed with parameter $\lambda = 1$. Show that

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log n} = 1 \right) = 1$$

Problem 31. Let $\{X_n : n \geq 1\}$ be independent $\mathcal{N}(0,1)$ random variables. Show that:

31.1. We have

$$\mathbf{P}\left(\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{\log n}} = \sqrt{2} \right) = 1.$$

31.2. It holds

$$\mathbf{P}\left(X_n > a_n \text{ i.o}\right) = \begin{cases} 0 & \text{if } \sum_n \mathbf{P}\left(X_1 > a_n\right) < \infty, \\ 1 & \text{if } \sum_n \mathbf{P}\left(X_1 > a_n\right) = \infty. \end{cases}$$

Problem 32. Construct an example to show that the convergence in distribution of X_n to X does not imply the convergence of the unique medians of the sequence X_n.

Problem 33. We consider cases of sequences $\{X_n;\, n \geq 1\}$.

33.1. Let $\{X_n;\, n \geq 1\}$ be independent, non-negative and identically distributed with infinite mean. Show that $\limsup_{n\to\infty} X_n/n = \infty$ almost surely.

33.2. Let $\{X_n\}$ be a stationary Markov chain on the positive integers with transition probabilities

$$p_{jk} = \begin{cases} \frac{j}{j+2} & \text{if } k = j+1 \\ \frac{2}{j+2} & \text{if } k = 1 \end{cases}$$

(1) Find the stationary distribution of the chain, and show that it has infinite mean.
(2) Show that $\limsup_{r\to\infty} X_r/r \leq 1$ almost surely.

Problem 34. Let $\{X_k : k \geq 1\}$ be independent and identically distributed with mean μ and finite variance σ^2. Let $\bar{X} = n^{-1}\sum_{k=1}^n X_k$ be the empirical mean. Show that

$$\frac{\sum_{k=1}^n \left(X_k - \mu\right)}{\left(\sum_{k=1}^n \left(X_k - \bar{X}\right)^2\right)^{1/2}} \xrightarrow{\text{(d)}} \mathcal{N}(0,1).$$

Problem 35. Let $\{X_k : k \geq 1\}$ be independent random variables such that

$$\mathbf{P}\left(X_n = n\right) = \mathbf{P}\left(X_n = -n\right) = \frac{1}{2n\log n}, \quad \text{and} \quad \mathbf{P}\left(X_n = 0\right) = 1 - \frac{1}{n\log n}.$$

Show that this sequence obeys the weak law but not the strong law, in the sense that $n^{-1}\sum_1^n X_i$ converges to 0 in probability but not almost surely.

Problem 36. Construct a sequence $\{X_k : k \geq 1\}$ of independent random variables with zero mean such that $n^{-1}\sum_{k=1}^n X_k \to -\infty$ almost surely, as $n \to \infty$.

Problem 37. Let N be a spatial Poisson process (the definition of such an object has been given in MA 532) with constant intensity λ in $\mathbb{R}^d$, where $d \geq 2$. Let S be the ball of radius r centered at zero. Show that $N(S)/|S| \to \lambda$ almost surely as $r \to \infty$, where $|S|$ is the volume of the ball.

Problem 38. The interval $[0,1]$ is partitioned into n disjoint sub-intervals with lengths $p_1, p_2, \ldots, p_n$. The so-called *entropy* of this partition is defined to be

$$h = -\sum_{i=1}^n p_i \log p_i.$$

Let $\{X_k : k \geq 1\}$ be independent random variables whose common distribution is $\mathcal{U}([0,1])$. Let $Z_m(i)$ be the number of the $X_1, X_2, \ldots, X_m$ which lie in the i-th interval of the partition above. Show that

$$R_m = \prod_{i=1}^{n} p_i^{Z_m(i)}$$

satisfies $m^{-1} \log R_m \to -h$ almost surely as $m \to \infty$.

Problem 39. Catastrophes occur at the times $\{T_k;\, k \geq 1\}$ where $T_k = X_1 + X_2 + \cdots + X_k$. We assume that the X_i's are independent identically distributed positive random variables. Let $N(t) = \max\{n : T_n \leq t\}$ be the number of catastrophes which have occurred by time t. Prove that if $\mathbf{E}[X_1] < \infty$ then $N(t) \to \infty$ and $N(t)/t \to 1/\mathbf{E}[X_1]$ as $t \to \infty$, almost surely.

Problem 40. Let $\{X_k : k \geq 1\}$ be independent identically distributed random variables taking values in the integers $\mathbb{Z}$ and having a finite mean. Show that the Markov chain $S = \{S_n;\, n \geq 1\}$ given by $S_n = \sum_{k=1}^{n} X_k$ is transient if $\mathbf{E}[X_1] \neq 0$. Note that S is a random walk.

Problem 41. Let $\{X_k : k \geq 1\}$ be independent identically distributed random variables and set $S_n = \sum_{k=1}^{n} X_k$. A function $\phi(x)$ is said to belong to the *upper class* if

$$\mathbf{P}(S_n > \phi(n)\sqrt{n} \text{ i.o.}) = 0.$$

A consequence of the law of the iterated logarithm is that $\sqrt{\alpha \log\log x}$ is in the upper class for all $\alpha > 2$. Use the first Borel-Cantelli lemma to prove the much weaker fact that $\phi(x) = \sqrt{\alpha \log x}$ is in the upper class for all $\alpha > 2$, in the special case when the X_i are independent $\mathcal{N}(0,1)$ variables.

4. Conditional expectation

Problem 42. Let X_1 and X_2 two independent random variables, both following a Poisson law with parameter λ. Let $Y = X_1 + X_2$. Compute

$$\mathbf{P}(X_1 = i | Y).$$

Problem 43. Let (X, Y) be a vector of $\mathbb{R}^2$, distributed uniformly over the unit disc. Compute the conditional density of X given Y.

Problem 44. Let (X, Y) be a couple of random variables with joint density

$$f(x,y) = 4y(x-y)\exp(-(x+y))\mathbf{1}_{0 \leq y \leq x}.$$

44.1. Compute $\mathbf{E}[X|Y]$.

44.2. Compute $\mathbf{P}(X < 1|Y)$.

Problem 45. We consider a head or tail type game, where the probability of getting head (resp. tail) is p (resp. $1-p$), with $0<p<1$. Player A throws the dice. He wins as soon as the number of heads exceeds the number of tails by a quantity of 2. He looses if the number of tails exceeds the number of heads by a quantity of 2. The game is stopped whenever A has won or lost.

45.1. Let E_n be the event: "the game is not over after $2n$ throws", $n \geq 1$. Show that $\mathbf{P}(E_n) = r^n$ where r is a real number to be determined.

45.2. Compute the probability that the player A wins and show that the game will stop a.s.

Problem 46. Let N be a random variable with values in $\{0,1,\dots,n\}$; we denote by $\alpha_k = \mathbf{P}(N=k)$. We consider a sequence $(\epsilon_n; n \geq 0)$ of independent random variables, whose common law is given by $\mathbf{P}(\epsilon_0 = 1) = p; \mathbf{P}(\epsilon_0 = 0) = q$, with $p+q=1, p>0, q>0$. We assume that N is independent of the family $(\epsilon_n; n \geq 1)$. We define a random variable X by the relation: $X = \sum_{k=1}^{N} \epsilon_k$.

46.1. Compute the law of X. Express

- $\mathbf{E}[X]$ in terms of $\mathbf{E}[N]$.
- $\mathbf{E}[X^2]$ in terms of $\mathbf{E}[N]$ and $\mathbf{E}[N^2]$.

46.2. Let $p' \in]0,1[$. Determine the law of N if we wish the conditional law of N given $X=0$ to be a binomial law $\mathcal{B}(n,p')$.

Problem 47. We consider the relations:

$$\mathbf{P}(X=0) = \frac{1}{3}; \qquad \mathbf{P}(X=2^n) = \mathbf{P}(X=-2^n) = \frac{2^{-n}}{3}; \qquad \forall n \geq 1. \tag{1}$$

47.1. Show that the relations (1) define a probability law for a random variable X.

47.2. Consider the following transition probability:

$$Q(0,.) = \frac{1}{2}(\delta_2 + \delta_{-2}), \qquad Q(x,.) = \frac{1}{2}(\delta_0 + \delta_{2x}), \qquad x \in \mathbb{R}^*,$$

where δ_a designates a Dirac measure in a. Let Y be a second real valued random variable such that the conditional law of Y given X is given by the transition probability Q. Show that $\mathbf{E}(Y|X) = X$ and that Y and X share the same law.

Problem 48. We note $\mathcal{B}_n$ the set of Borel sets of $\mathbb{R}^n$, and let $\mathcal{S}_n$ be the set of symmetric Borel sets A of $\mathbb{R}^n$, i.e. $-A = A$.

48.1. Show that $\mathcal{S}_n$ is a sub σ-algebra of $\mathcal{B}_n$ and that a random variable Y is $\mathcal{S}_n$-measurable if and only if $Y(-x) = Y(x)$.

48.2. We say that a probability measure P over $(\mathbb{R}^n, \mathcal{B}_n)$ is is symmetric if $\mathbf{P}(A) = \mathbf{P}(-A)$ for all A lying in $\mathcal{B}_n$. Show that if ϕ is a real valued integrable random variable defined on $(\mathbb{R}^n, \mathcal{B}_n, P)$, we have: $\mathbf{E}[\phi|\mathcal{S}_n](x) = \frac{1}{2}(\phi(x) + \phi(-x))$.

48.3. We assume $n=1$ and we denote by X the identity application of $\mathbb{R}$ onto $\mathbb{R}$. Determine $\mathbf{E}[\phi||X||]$ and $\mathbf{E}[\phi|X^2]$.

Problem 49. Let X and Y two real valued and independent random variables, with uniform law on $[0,1]$. We set $U = \inf\{X,Y\}$ and $V = \sup\{X,Y\}$. Compute $\mathbf{E}[U|V]$ and the best prediction of U by an affine function of V.

Problem 50. Let $G \in \mathcal{G}$. Show that

$$\mathbf{P}(G|A) = \frac{\int_G \mathbf{P}(A|\mathcal{G})dP}{\int_\Omega \mathbf{P}(A|\mathcal{G})dP}.$$

This can be seen as a general version of Baye's formula.

Problem 51. Let X and Y be two random variables such that $X \leq Y$, and $a > 0$.

51.1. Show that $\mathbf{E}[X|\mathcal{F}] \leq \mathbf{E}[Y|\mathcal{F}]$.

51.2. Show that

$$\mathbf{P}(|X| \geq a|\mathcal{F}) \leq \frac{\mathbf{E}[X^2|\mathcal{F}]}{a^2}.$$

Problem 52. Let X_1 and X_2 be two random variables such that $X_1 = X_2$ on $B \in \mathcal{F}$. Show that

$$\mathbf{E}[X_1|\mathcal{F}] = \mathbf{E}[X_2|\mathcal{F}] \qquad \text{a.s on } B.$$

Problem 53. Give an example on $\Omega = \{a,b,c\}$ for which

$$\mathbf{E}\left[\mathbf{E}[X|\mathcal{F}_1]|\mathcal{F}_2\right] \neq \mathbf{E}\left[\mathbf{E}[X|\mathcal{F}_2]|\mathcal{F}_1\right].$$

Problem 54. Let X and Y be two random variables.

54.1. Show that if X and Y are independent, then $\mathbf{E}[X|Y] = \mathbf{E}[X]$.

54.2. Give an example of random variables with values in $\{-1,0,1\}$ such that X and Y are not independent, in spite of the fact that $\mathbf{E}[X|Y] = \mathbf{E}[X]$.

Problem 55. Let $n \geq 1$ be a fixed integer and p_1, p_2, p_3 three positive real numbers satisfying $p_1 + p_2 + p_3 = 1$. We set:

$$p_{i,j} = n!\, \frac{p_1^i\, p_2^j\, p_3^{n-i-j}}{i!\, j!\, (n-i-j)!}$$

whenever $i + j \leq n$, and $p_{i,j} = 0$ if $i + j > n$.

55.1. Show that there exists a couple of random variables (X,Y) such that $\mathbf{P}(X = i, Y = j) = p_{i,j}$.

55.2. Determine the law of X, the law of Y and the law of Y given X, expressed as a conditional regular law.

55.3. Compute $\mathbf{E}[XY]$ thanks to the conditional regular law introduced in the previous question 55.2.

Problem 56. Let $X_1, X_2, \ldots, X_n$, n be some real valued integrable random variables, independent and equally distributed. We set $m = \mathbf{E}[X_1]$ and $S_n = \sum_{i=1}^n X_i$.

56.1. Compute $\mathbf{E}[S_n|X_i]$ for all $i, 1 \leq i \leq n$.

56.2. Compute $\mathbf{E}[X_i|S_n]$ for all $i, 1 \leq i \leq n$.

56.3. We assume now that $n = 2$ and that the random variables X_i have a common density φ. Compute the conditional density of X_i given S_2. Give a specific expression whenever the law of each X_i is an exponential law.

Problem 57. The random vector (X, Y) has a density

$$f_{X,Y}(x, y) = \frac{x}{\sqrt{2\pi}} \exp(-\frac{1}{2}x(y + x))\mathbf{1}_{\{x>0\}}\mathbf{1}_{\{y>0\}}.$$

Determine the conditional distribution of Y given X, expressed as a conditional regular law.

Problem 58. Let X and Y be two real valued random variables, such that Y follows an exponential law. We assume that given Y, X is distributed according to a Poisson law with parameter Y (given as a conditional regular law).

58.1. Compute the law of the couple (X, Y), the law of X, and the law of Y given X as a conditional regular law.

58.2. Show that $\mathbf{E}[(Y - X)^2] = 1$, conditioning first with respect to Y, then integrating with respect to Y.

Problem 59. Let $X_1, X_2, \ldots, X_n$ be n real valued independent random variables, admitting a common density p.

59.1. Show that for all $i \neq j, \mathbf{P}(X_i = X_j) = 0$. In the following we set $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ for the sequence $\{X_1, X_2, \ldots, X_n\}$ arranged in increasing order:

$$X_{(1)} < X_{(2)} < \cdots < X_{(n)}.$$

59.2. Show that $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ admits a density of the form:

$$f(x_1, x_2, \ldots, x_n) = n!\, p(x_1)p(x_2)\cdots p(x_n)\mathbf{1}_{\{x_1<x_2<\cdots<x_n\}}$$

59.3. We assume now that the common law of the random variables X_i is the uniform law on $[a, b]$.

(1) Determine the density of $(X_{(1)}, X_{(n)})$.
(2) We set $\mu_n(a, b; x_1, x_2, \ldots, x_n)$ for the density of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. Show that conditionally on $X_{(1)}$ and $X_{(n)}$ the vector $X_{(2)}, X_{(3)}, \ldots, X_{(n-1)}$ admits the CRL given by $\mu_{n-2}(X_1, X_n; x_2, x_3, \ldots, x_{n-1})$. Deduce that

$$\left(\frac{X_{(2)} - X_{(1)}}{X_{(n)} - X_{(1)}}, \ldots, \frac{X_{(n-1)} - X_{(1)}}{X_{(n)} - X_{(1)}}\right)$$

is a random variable independent of $(X_{(1)}, X_{(n)})$ and possesses a density $\mu_{n-1}(0, 1; \cdot)$.

5. Discrete time martingales

Problem 60. Let $\{X_n; n \geq 1\}$ be a martingale with respect to a filtration $\mathcal{G}_n$, and let

$$\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}.$$

Show that $\mathcal{F}_n \subset \mathcal{G}_n$ and that X_n is a $\mathcal{F}_n$-martingale.

Problem 61. We say that $f : \mathbb{R}^d \to \mathbb{R}$ is a super-harmonic function whenever $\Delta f \leq 0$. These functions satisfy:

$$f(x) \geq \int_{\partial B(x,r)} f(y) d\pi(y),$$

where $\partial B(x,r) = \{y; |x-y| = r\}$ is the boundary of the ball centered at x with radius r, and π is the surface measure of this boundary. Let $f \geq 0$ be a super-harmonic function on $\mathbb{R}^d$, and let $\{\xi_n; n \geq 1\}$ be a sequence of iid random variables, with common uniform law on $\partial B(0,1)$. We set $S_n = \xi_n + S_{n-1}$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Problem 62. Let $\{\xi_n; n \geq 1\}$ be a sequence of independent random variables such that $\xi_j \in L^1(\Omega)$ and $\mathbf{E}[\xi_j] = 0$. We set

$$X_n = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \xi_{i_1} \cdots \xi_{i_k}.$$

Show that X is a martingale.

Problem 63. Let $\{X_n; n \geq 1\}$ and $\{Y_n; n \geq 1\}$ be two sub-martingales with respect to $\mathcal{F}_n$. Show that $X_n \vee Y_n$ is a sub-martingale.

Problem 64. Let $\{Y_n; n \geq 1\}$ be a iid sequence of positive random variables such that $\mathbf{E}[Y_j] = 1$, $\mathbf{P}(Y_j = 1) < 1$ and $\mathbf{P}(Y_j = 0) = 0$. We set

$$X_n = \Pi_{j \leq n} Y_j.$$

64.1. Show that X is a martingale.

64.2. Show that $\lim_{n\to\infty} X_n = 0$ a.s

Problem 65. We wish to study a branching process defined in the following way: let $\{\xi_i^n;\ i, n \geq 1\}$ be a sequence of iid random integer valued random variables. We set $Z_0 = 1$ and pour $n \geq 0$,

$$Z_{n+1} = \left(\sum_{i=1}^{Z_n} \xi_i^{n+1} \right) \mathbf{1}_{(Z_n > 0)}.$$

This process is called *Galton Watson process*, and represents the number of living individuals at each generation in various biological models. We set

$$\mathcal{F}_n = \sigma\{\xi_i^m;\ 1 \leq m \leq n,\ i \geq 1\},$$

and $\mu = \mathbf{E}[\xi_i^n]$.

65.1. Show that $\frac{Z_n}{\mu^n}$ is a $\mathcal{F}_n$-martingale.

65.2. Show that $\frac{Z_n}{\mu^n}$ converges a.s to a random variable Z_∞.

65.3. We assume now that $\mu < 1$.

(1) Show that $\mathbf{P}(Z_n > 0) \leq \mathbf{E}[Z_n]$.
(2) Show that Z_n converges in probability to 0.
(3) Show that $Z_n = 0$ for n large enough.

65.4. We assume now that $\mu = 1$ and $\mathbf{P}(\xi_i^n = 1) < 1$.

(1) Show that Z_n converges a.s to a random variable Z_∞.
(2) Suppose that $\mathbf{P}(Z_\infty = k) > 0$ for $k \geq 0$. Show that there exists $N > 0$ such that
$$\mathbf{P}(Z_n = k \text{ for all } n \geq N) > 0.$$
(3) Show that $\mathbf{P}(Z_n = k \text{ for all } n \geq N) = 0$ for all $k > 0$.
(4) Deduce that $Z_\infty = 0$.

65.5. Eventually we assume that $\mu > 1$, and we will show that
$$\mathbf{P}(Z_n > 0 \text{ for all } n \geq 0) > 0.$$
To this aim, we set $p_k = \mathbf{P}(\xi_i^n = k)$, and set
$$\phi(s) = \sum_{k=0}^{\infty} p_k s^k, \qquad s \in [0,1].$$
Namely, ϕ is the moment generating function of ξ_i^n.

(1) Show that ϕ is increasing, convex, and that $\lim_{s \to 1} \phi'(s) = \mu$.
(2) Let $\theta_m = \mathbf{P}(Z_m = 0)$. Show that $\theta_m = \phi(\theta_{m-1})$.
(3) Invoking the fact that $\phi'(1) > 1$, show that there exists at least one root for the equation $\phi(x) = x$ in $[0,1)$. Let ρ be the smallest of those roots.
(4) Show that ϕ is strictly convex, and deduce that ρ is the unique root of $\phi(x) = x$ in $[0,1)$.
(5) Show that the extinction probability is such that
$$\mathbf{P}(Z_n = 0 \text{ for some } n \geq 0) = \rho < 1.$$

65.6. Galton and Watson were interested in family name survivals. Suppose that each family has exactly three children, and that the gender distribution is uniform. In 19^{th} century England, only males could keep their family names. Compute the survival probability in this context.

Problem 66. Let $\{Y_n; n \geq 1\}$ be a sequence of independent random variables, with common Gaussian law $\mathcal{N}(0, \sigma^2)$, where $\sigma > 0$. We set $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ and $X_n = \sum_{i=1}^n Y_i$. Recall that:
$$\mathbf{E}[\exp(uY_1)] = \exp\left(\frac{u^2\sigma^2}{2}\right).$$
We also set, for all $u \in \mathbb{R}^*$,
$$Z_n^u = \exp\left(uX_n - \frac{1}{2}nu^2\sigma^2\right).$$

66.1. Show that $\{Z_n^u; n \geq 1\}$ is a $\mathcal{F}_n$-martingale for all $u \in \mathbb{R}^*$.

66.2. We wish to study the almost sure convergence of Z_n^u for $u \in \mathbb{R}^*$.

(1) Show that for all $u \in \mathbb{R}^*$, Z_n^u converges almost surely.
(2) Show that

$$K_n \equiv \frac{1}{n}\left(uX_n - \frac{1}{2}nu^2\sigma^2\right)$$

converges almost surely, and determine its limit.
(3) Find the almost sure limit of Z_n^u for $u \in \mathbb{R}^*$.

66.3. We now study the L^1-convergence of Z_n^u, for $u \in \mathbb{R}^*$.

(1) Find $\lim_{n\to\infty} \mathbf{E}[Z_n^u]$.
(2) Is the martingale Z_n^u converging in L^1?

Problem 67. At time 1, an urn contains 1 green ball and 1 blue ball. A ball is drawn, and replaced by 2 balls of the same color as the one which has been drawn. This gives a new composition at time 2. This procedure is then repeated successively. We set Y_n for the number of green balls at time n, and write $X_n = \frac{Y_n}{n+1}$ for the proportion of green balls at time n. We also set $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

67.1. Show that $\mathbf{E}[Y_{n+1}|\mathcal{F}_n] = (Y_n + 1)X_n + Y_n(1 - X_n)$.

67.2. Show that $\{X_n; n \geq 1\}$ is a $\mathcal{F}_n$-martingale, which converges almost surely to a random variable U.

67.3. By means of the dominated convergence theorem, show that for all $k \geq 1$, we have $\lim_{n\to\infty} \mathbf{E}[X_n^k] = \mathbf{E}[U^k]$.

67.4. Fix $k \geq 1$. We set, for $n \geq 1$,

$$Z_n = \frac{Y_n(Y_n+1)\dots(Y_n+k-1)}{(n+1)(n+2)\dots(n+k)}.$$

(1) Let us define the random variables $\mathbf{1}_{\{Y_{n+1}=Y_n\}}$ and $\mathbf{1}_{\{Y_{n+1}=Y_n+1\}}$. Relying on these quantities, show that $\{Z_n; n \geq 1\}$ is a $\mathcal{F}_n$-martingale.
(2) Express the almost sure limit of Z_n as a function of the random variable U.
(3) Compute the value of $\mathbf{E}[U^k]$.
(4) Show that these moments are those of the law $\mathcal{U}([0,1])$.

Problem 68. Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to a filtration $\mathcal{F}_n$. We assume that there exists a constant $M > 0$ such that for all $n \geq 1$

$$\mathbf{E}\left[|X_n - X_{n-1}| \,\middle|\, \mathcal{F}_{n-1}\right] \leq M \quad a.s.$$

68.1. Show that if $(V_n)_{n\geq 1}$ is a predictable (i.e V_n is $\mathcal{F}_{n-1}$−measurable) process taking positive values, then we have

$$\sum_{n=1}^{\infty} V_n \mathbf{E}\left[|X_n - X_{n-1}|\middle|\mathcal{F}_{n-1}\right] \leq M \sum_{n=1}^{\infty} V_n.$$

68.2. Let ν be an integrable stopping time. Show that X_ν is integrable, and that $X_{\nu\wedge p}$ converges to X_ν in L^1. Deduce that $\mathbf{E}(X_\nu) = \mathbf{E}(X_0)$.
Hint : write

$$X_\nu - X_{\nu\wedge p} = \sum_{n=1}^{\infty} \mathbf{1}_{\{\nu\wedge p < n \leq \nu\}} (X_n - X_{n-1}) .$$

68.3. Show that if $\nu_1 \leq \nu_2$ are two stopping times with ν_2 integrable, then $\mathbf{E}[X_{\nu_2}] = \mathbf{E}[X_{\nu_1}]$.

Problem 69. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables with a common law given by $\mathbf{P}(Y_n = 1) = p = 1 - \mathbf{P}(Y_n = -1) = 1 - q$. We define $(S_n)_{n\in\mathbb{N}}$ by $S_0 = 0$ and $S_n = \sum_{k=1}^n Y_k$.

69.1. We assume that $p = q = \frac{1}{2}$. We set $T_a = \inf\{n \geq 0,\ S_n = a\} \quad (a \in \mathbb{Z}^*)$. Show that $\mathbf{E}(T_a) = +\infty$.

69.2. Let $T = T_{a,b} = \inf\{n \geq 0,\ S_n = -a \text{ or } S_n = b\} \quad (a, b \in \mathbb{N})$. Using the value of $\mathbf{E}(S_T)$, compute the probability of the event $(S_T = -a)$.

69.3. Show that $Z_n = S_n^2 - n$ is a martingale, and from the value of $\mathbf{E}(Z_T)$ compute $\mathbf{E}(T)$.

69.4. We assume that $p > q$ and we set $\mu = \mathbf{E}(Y_k)$. Show that

$$X_n = S_n - n\mu \quad \text{and} \quad U_n = \left(\frac{q}{p}\right)^{S_n}$$

are martingales. Deduce the value of $\mathbf{P}(S_T = -a)$ and $\mathbf{E}(T)$.

6. Discrete models in finance

Problem 70. This problem is concerned with the Cox, Ross and Rubinstein model: we consider a unique risky asset whose price at time n is called R_n, as well as a non risky asset with price $S_n = (1+r)^n$. We assume the following about R_n: between time n and $n+1$ the relative variation of price is either a or b, with $-1 < a < b$. Otherwise stated:

$$R_{n+1} = (1+a)R_n \quad \text{or} \quad R_{n+1} = (1+b)R_n, \quad n = 0, \ldots, N-1.$$

The natural space for all possible results is thus $\Omega = \{1+a, 1+b\}^N$, and we also consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$. The set Ω is equipped with a probability $\mathbf{P}$ such that all singletons of Ω have a non zero probability. Set $T_n = \frac{R_n}{R_{n-1}}$, and note that $\mathcal{F}_n = \sigma(T_1, \ldots, T_n)$.

70.1. Show that the actualized price $\tilde{R}_n$ is a $\mathbf{P}$-martingale if and only if $\mathbf{E}[T_{n+1}|\mathcal{F}_n] = 1 + r$.

70.2. Deduce that, in order to have a viable market, the rate r should satisfy the condition $r \in (a, b)$.

70.3. Give an example of possible arbitrage whenever $r \notin (a, b)$.

70.4. We assume in the remainder of the problem that $r \in (a,b)$, and we set $p = \frac{b-r}{b-a}$. Show that $\tilde{R}_n$ is a martingale under $\mathbf{P}$ if and only if the random variables T_j are independent, equally distributed, with a common law given by:

$$\mathbf{P}(T_j = 1+a) = p = 1 - \mathbf{P}(T_j = 1+b).$$

Then prove that the market is complete.

70.5. Let C_n (resp. P_n) be the value at time n, of a European call (resp. put).

(1) Show that

$$C_n - P_n = R_n - K(1+r)^{-(N-n)}.$$

This general relation is known as *call-put parity*.

(2) Show that C_n can be written as $C_n = c(n, R_n)$, where c is a function which will be expressed thanks to the constants K, a, b, p.

70.6. Show that a perfect hedging strategy for a call is defined by a quantity $H_n = \Delta(n, R_{n-1})$ of risky asset which should be held at time n, where Δ is a function which can be expressed in terms of c.

Problem 71. In this problem we consider a *multinomial* Cox, Ross and Rubinstein model: the unique risky asset has a price R_n at time n, and the non risky asset price is given by $S_n = (1+r)^n$. We assume the following for the risky asset price: between time n and $n+1$ the relative variation of price belongs to the set $\{a_1, a_2, \dots, a_k\}$, with $k \geq 3$ and $-1 < a_1 < a_2 < \dots < a_k$. Otherwise stated:

$$R_{n+1} = (1+a_j)R_n \quad \text{with} \quad j \in \{1, 2, \dots, k\}, \quad n = 0, \dots, N-1.$$

The natural space for all possible results is thus $\Omega = \{1+a_1, \dots, 1+a_k\}^N$, and we also consider $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathcal{F}_n = \sigma(R_1, \dots, R_n)$. The set Ω is equipped with a probability $\mathbf{P}$ such that all singletons of Ω have a non zero probability. Set $T_n = \frac{R_n}{R_{n-1}}$, and note that $\mathcal{F}_n = \sigma(T_1, \dots, T_n)$. We set

$$p_{n,j} = \mathbf{P}\left(T_n = 1+a_j\right), \quad j \in \{1, 2, \dots, k\}, \quad n = 0, \dots, N-1.$$

71.1. Show that the actualized price $\tilde{R}_n$ is a $\mathbf{P}$-martingale if and only if $\mathbf{E}[T_{n+1}|\mathcal{F}_n] = 1+r$.

71.2. Deduce that, in order to have a viable market, the rate r should satisfy the condition $r \in [a_1; a_k]$.

71.3. Give an example of possible arbitrage whenever $r < a_1$.

71.4. We assume in the remainder of the problem that

$$r = \frac{1}{k} \sum_{j=1}^{k} a_j.$$

Let $\mathcal{Q}$ be the set of probability measures $\mathbf{Q}$ on Ω satisfying:

(i) Under $\mathbf{Q}$, the family $\{T_n; n \leq N-1\}$ is a family of i.i.d random variables.
(ii) $\tilde{R}_n$ is a $\mathbf{Q}$-martingale.

(1) Let $\mathbf{Q}^{(1)}$ be the probability on Ω defined by: the family of random variables $\{T_n; n \leq N-1\}$ is a family of i.i.d random variables of common law given by:

$$Q^{(1)}(T_n = 1 + a_j) = \frac{1}{k}, \quad j \in \{1, 2, \ldots, k\}.$$

Show that $Q^{(1)} \in \mathcal{Q}$.

(2) Show that $\mathcal{Q}$ is an infinite set.

(3) Show that the market is incomplete.

71.5. We now work under the probability $\mathbf{Q}^{(1)}$. Let C_n be the value of a European call with strike K and maturity N. Show that C_n can be written under the form $C_n = c(n, R_n)$, where c is a function which will be expressed thanks to $K, a_1, \ldots, a_k$. Note that a multinomial law can be used here. This law can be defined as follows: we consider an urn with a proportion p_j of balls of type j, for $j \in \{1, \ldots, k\}$, with $\sum_{j=1}^k p_j = 1$. We draw n times from this urn and we call X_j the number of balls of type j obtained in this way. Then for for any tuple of integers $(n_1, \ldots, n_j)$ such that $\sum_{j=1}^k n_j = n$, we have

$$\mathbf{P}(X_1 = n_1, \ldots, X_k = n_k) = \frac{n!}{\prod_{j=1}^k n_j!} \prod_{j=1}^k p_j^{n_j}.$$

La law of the vector $(X_1, \ldots, X_k)$ ia called multinomial law with parameters $(n, k, p_1, \ldots, p_k)$.

7. Gaussian vectors and CLT

Problem 72. Let ϵ be a Rademacher random variable, that is:

$$\mathbf{P}(\epsilon = 1) = \mathbf{P}(\epsilon = -1) = 1/2.$$

Assume that ϵ is independent of X, where $X \sim \mathcal{N}_1(0, 1)$.

72.1. Show that the law of ϵX is still Gaussian.

72.2. Show that $X + \varepsilon X$ is not a Gaussian variable. Deduce that the random vector $(X, \varepsilon X)$ is not a Gaussian vector.

Problem 73. Let X, Y be two independent standard Gaussian random variables.

73.1. Show that $\frac{X}{Y}$ is well-defined, and is distributed according to a Cauchy law.

73.2. If $t \geq 0$, compute $\mathbf{P}(|X| \leq t|Y|)$.

Problem 74. If (X, Y) is a centered Gaussian vector in $\mathbb{R}^2$ with $\mathbf{E}[X^2] = \mathbf{E}[Y^2] = 1$ and if $\mathbf{E}[XY] = r$ with $r \in (-1, 1)$, compute $\mathbf{P}(XY \geq 0)$. *Hint:* one can prove and use the following claim: $(X, Y) = (X,\ sX + \sqrt{1 - s^2} Z)$ with $X, Z \sim \mathcal{N}(0, 1)$ independent and $s \in (0, 1)$ to be determined. Then we invoke the result shown in Problem 73.

Problem 75. Let $X, Y \sim \mathcal{N}(0, 1)$ be two independent random variables. For all $a \in (-1, 1)$, show that:

$$\mathbf{E}[\exp(aXY)] = \mathbf{E}\left[\exp\left(\frac{a}{2} X^2\right)\right] \mathbf{E}\left[\exp\left(-\frac{a}{2} Y^2\right)\right].$$

Problem 76. Let X and Y two independent standard Gaussian random variables $\mathcal{N}(0,1)$. We set $U = X^2 + Y^2$ and $V = \frac{X}{\sqrt{U}}$. Show that U and V are independent, and compute their law.

Problem 77. Let A be the matrix defined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

77.1. Show that there exist a centered Gaussian vector G with covariance matrix A. The coordinates of G are denoted by X, Y and Z.

77.2. Is G a random variable with density? Compute the characteristic function of G.

77.3. Characterize the law of $U = X + Y + Z$.

77.4. Show that $(X - Y, X + Z)$ is a Gaussian vector.

77.5. Determine the set of random variables $\xi = aX + bY + cZ$, independent of U.

Problem 78. Let Q be a positive definite quadratic form defined on $\mathbb{R}^n$. We introduce a function f given as

$$f(x) = \lambda \exp\left(-\frac{Q(x)}{2}\right), \qquad x \in \mathbb{R}^n.$$

78.1. According to Q, compute the unique value λ such that f is a density. *Hint:* show that f can be seen as the density of a Gaussian vector.

78.2. Application: $n = 2$, $Q(x, y) = 3x^2 + y^2 + 2xy$.

Problem 79. Let $X = (X_1, \ldots, X_n)$ be a centered Gaussian vector with covariance matrix Id_n.

79.1. Show that the random vector $(X_1 - \bar{X}, \ldots, X_n - \bar{X})^*$ is independent of $\bar{X}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

79.2. Deduce that the random variables $\bar{X}$ and $W = \max_{1\le i\le n} X_i - \min_{1\le i\le n} X_i$ are independent. Why is this result (somewhat) surprising?

Problem 80. A restaurant can serve 75 meals. In practice, it has been established that 20% of customers with a reservation do not show up.

80.1. The restaurant owner has accepted 90 reservations. What is the probability that more than 65 persons will come?

80.2. What is the maximal number of reservations which can be accepted if we wish to serve all customers with probability ≥ 0.9?

Problem 81. Let $(X_n; n \geq 1)$ be a sequence of i.i.d $\mathbb{R}^k$-valued random variables, which are assumed to be square integrable. In the sequel $(\xi_n; n \geq 1)$ designates a sequence of i.i.d bounded real-valued random variables. We assume that $(X_n; n \geq 1)$ is independent of $(\xi_n; n \geq 1)$ and also that either X_1 or ξ_1 is centered. We set

$$Y_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \xi_i X_i.$$

Show that Y_n converges in distribution as n goes to infinity. Characterize the limiting law.

Problem 82. The aim of this problem is to show that the Laplace transform characterizes probability laws on $\mathbb{R}_+$. To this aim, for all $t > 0$ and $x > 0$ we set,

$$a_n(x,t) = \int_0^{n/t} \frac{y^{n-1} x^n}{(n-1)!} e^{-yx} dy.$$

82.1. Invoking the law of large numbers (resp. central limit theorem), show that

$$\lim_{n \to \infty} a_n(x,t) = \begin{cases} \mathbf{1}_{\{x>t\}} & \text{if } x \neq t \\ \frac{1}{2} & \text{if } x = t. \end{cases}$$

82.2. Let X be a random variable with values in $\mathbb{R}_+$. We set $G(\theta) = E[e^{-\theta X}]$.

(1) Using Question 82.1, show that:

$$\lim_{n \to \infty} (-1)^n \int_0^{n/t} \frac{y^{n-1}}{(n-1)!} \frac{d^n G}{dy^n}(y) dy = \frac{1}{2} P(X = t) + P(X > t).$$

(2) Deduce that G characterizes the distribution of X.

Problem 83. Let X and Y two real valued i.i.d random variables. We assume that $\frac{X+Y}{\sqrt{2}}$ has the same law as X and Y. We also suppose that this common law admits a variance, denoted by σ^2.

83.1. Show that X is centered random variable.

83.2. Show that if X_1, X_2, Y_1 and Y_2 are independent random variables having the same law as X, then $\frac{1}{2}(X_1 + X_2 + Y_1 + Y_2)$ has the same law as X.

83.3. Applying the central limit theorem, show that X is distributed as $\mathcal{N}(0, \sigma^2)$.

Problem 84. The aim of this problem is to give an example of application for the multidimensional central limit theorem. Let $(Y_i;\ i \geq 1)$ be a sequence of i.i.d real valued random variables. We will denote by F common cumulative distribution function and $\hat{F}_n$ the empirical cumulative distribution function for the n-sample $(Y_1, \ldots, Y_n)$:

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{Y_i \leq x\}}, \quad x \in \mathbb{R}.$$

84.1. Let x a fixed real number. Show that:

- $\hat{F}_n(x)$ converges a.s. to $F(x)$, when $n \to \infty$;

- $\sqrt{n}(\hat{F}_n(x) - F(x))$ converges in law, when $n \to \infty$, to a centered Gaussian random variable with variance $F(x)(1 - F(x))$.

84.2. We will generalize this result to a multidimensional setting. Let $x_1, x_2, \ldots, x_d$ be a sequence of real numbers such that $x_1 < x_2 < \cdots < x_d$, and X_n be the random vector in $\mathbb{R}^d$, with coordinates $X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}$ where:

$$X_n^{(i)} = \mathbf{1}_{\{Y_n \leq x_i\}}; \quad 1 \leq i \leq d,$$

for all $n \geq 1$. Show that:

$$\left(\sqrt{n}(F_n(x_1) - F(x_1)), \ldots, \sqrt{n}(F_n(x_d) - F(x_d))\right)$$

converges in law, when $n \to \infty$, to a centered Gaussian vector for which we will compute the covariance matrix.