Probability preliminaries

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Stochastic calculus - MA598

Outline

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First definitions

Probability space: $(\Omega, \mathcal{F}, \mathbf{P})$ with

- Ω set
- $\mathcal{F}$ generic σ-algebra
- $\mathbf{P}$ probability measure

Hypothesis: We assume that $\mathbf{P}$ is complete, i.e

$$A \in \mathcal{F} \text{ such that } \mathbf{P}(A) = 0, \text{ and } B \subset A \quad \Longrightarrow \quad B \in \mathcal{F} \text{ and } \mathbf{P}(B) = 0$$

Remark: One can always complete a probability space

Simple examples

Rolling 2 dice:

- $\Omega = \{1,2,3,4,5,6\}^2$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathbf{P}(A) = \frac{|A|}{36}$

Uniform law on $[0,1]$:

- $\Omega = [0,1]$
- $\mathcal{F} = \mathcal{B}([0,1])$
- $\mathbf{P} = \lambda$, Lebesgue measure

Gaussian law on $\mathbb{R}$:

- $\Omega = \mathbb{R}$
- $\mathcal{F} = \mathcal{B}(\mathbb{R})$
- $\mathbf{P}(A) = \frac{1}{(2\pi)^{1/2}} \int_A e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$, for $A \in \mathcal{F}$

Important example for stochastic processes

Proposition 1.

Let $\Omega = \mathcal{C}([0,\infty);\mathbb{R}^m)$. We set:

$$d(f,g) = \sum_{n\geq 1} \frac{\|f-g\|_{\infty,n}}{2^n\left(1+\|f-g\|_{\infty,n}\right)},$$

where

$$\|f-g\|_{\infty,n} = sup\left\{|f_t - g_t|;\ t\in[0,n]\right\}.$$

Then Ω is a complete separable metric space.

Following chapter: We construct the Wiener measure on Ω

Independence

Independence of r.v: Let $(X_j)_{j\in J}$ r.v, with values in $\mathbb{R}^n$.
We say that these r.v are independent if for all $m \geq 2$:

- For all $j_1, \ldots, j_m \in J$, the r.v $(X_{j_1}, \ldots, X_{j_m})$ are $\perp\!\!\!\perp$
- Otherwise stated: for all $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^n)$ we have

$$\mathbf{P}\left(X_{j_1} \in A_1, \ldots, X_{j_m} \in A_m\right) = \prod_{k=1}^{m} \mathbf{P}\left(X_{j_k} \in A_k\right)$$

Independence of σ-algebras: Let $(\mathcal{F}_j)_{j\in J}$ σ-algebras, $\mathcal{F}_j \subset \mathcal{F}$.
We say that those σ-algebras are independent if for all $m \geq 2$:

- For all $j_1, \ldots, j_m \in J$, the σ-algebras $(\mathcal{F}_{j_1}, \ldots, \mathcal{F}_{j_m})$ are $\perp\!\!\!\perp$
- Otherwise stated: for all $B_1 \in \mathcal{F}_{j_1}, \ldots, B_m \in \mathcal{F}_{j_m}$ we have

$$\mathbf{P}\left(\bigcap_{k=1}^{m} B_k\right) = \prod_{k=1}^{m} \mathbf{P}\left(B_k\right)$$

π-systems and λ-systems

π-system: Let $\mathcal{P}$ family of sets in Ω. $\mathcal{P}$ is a π-system if:

$$A, B \in \mathcal{P} \quad \Longrightarrow \quad A \cap B \in \mathcal{P}$$

λ-system: Let $\mathcal{L}$ family of sets in Ω. $\mathcal{L}$ is a λ-system if:

1. $\Omega \in \mathcal{L}$
2. If $A \in \mathcal{L}$, then $A^c \in \mathcal{L}$
3. If for $j \geq 1$ we have:
    - $A_j \in \mathcal{L}$
    - $A_j \cap A_i = \emptyset$ if $j \neq i$

   Then $\cup_{j \geq 1} A_j \in \mathcal{L}$

Dynkin's π-λ lemma

Lemma 2.

Let $\mathcal{P}$ and $\mathcal{L}$ such that:

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mathcal{P} \subset \mathcal{L}$

Then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Application of Dynkin's lemma

Proposition 3.

Let:

- $X_1, \dots, X_n$ r.v, with values in $\mathbb{R}^m$.
- $X \equiv (X_1, \dots, X_n) \in \mathbb{R}^{m \times n}$.
- $\mu_{X_j} = \mathcal{L}(X_j)$ and $\mu_X = \mathcal{L}(X)$.

Then the following two statements are equivalent:

1. $X_1, \dots, X_n$ are independent
2. $\mu_X = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n}$ on $\mathcal{B}(\mathbb{R}^{m \times n})$

Proof

Definition of two systems: We set

$$\mu_1 = \mu_X, \quad \text{and} \quad \mu_2 = \mu_{X_1} \otimes \cdots \otimes \mu_{X_n},$$

and

$$\begin{aligned} \mathcal{P} &\equiv \left\{ A \in \mathcal{B}(\mathbb{R}^{m\times n});\ A = A_1 \times \cdots \times A_n,\ \text{where } A_j \in \mathcal{B}(\mathbb{R}^m) \right\} \\ \mathcal{L} &\equiv \left\{ B \in \mathcal{B}(\mathbb{R}^{m\times n});\ \mu_1(B) = \mu_2(B) \right\}. \end{aligned}$$

Application of Dynkin's lemma: We have

- $\mathcal{P}$ is a π-system
- $\mathcal{L}$ is a λ-system
- $\mu_1(C) = \mu_2(C)$ for all $C \in \mathcal{P}$

Thus $\sigma(\mathcal{P}) \subset \mathcal{L}$, and $\sigma(\mathcal{P}) = \mathcal{B}(\mathbb{R}^{m\times n})$

Outline

Aim

Situation: We consider

- $\{\mu_k;\ k \geq 1\}$ sequence of probability measures on $\mathbb{R}$

We try to define:

$$\mu = \bigotimes_{k=1}^{\infty} \mu_k,$$

on a probability space

$$\Omega \equiv \prod_{k=1}^{\infty} \Omega_k.$$

Cylinder sets

Recall: We consider

- $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$ family of probability spaces
- $\Omega \equiv \prod_{k=1}^{\infty} \Omega_k$

Definition 4.

Let $A \subset \Omega$. We say that A is cylindrical
if there exists $k \geq 0$ and $0 \leq n_1 < \cdots < n_k$ such that

$$A = \{\omega \in \Omega; \, \omega_{n_1} \in A_1, \ldots, \omega_{n_k} \in A_k\}, \quad \text{where} \quad A_j \in \mathcal{F}_{n_j}$$

Interpretation:
A cylindrical set only involves a finite number of coordinates

product σ-algebra on Ω: $\mathcal{F} \equiv \sigma(\mathcal{C})$, with $\mathcal{C} \equiv$ cylindrical sets.

Product measure

Theorem 5.

Let:

- $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k)$ family of probability spaces
- $(\Omega, \mathcal{F})$ product space

Then there exists a unique probability $\mathbf{P}$ on $(\Omega, \mathcal{F})$ such that:

$$\mathbf{P}(A) = \prod_{j=1}^{k} \mathbf{P}_{n_j}(A_j), \quad \text{for all} \quad A \in \mathcal{C}$$

Sequence of independent random variables

Theorem 6.

Let:

- $\{\mu_k;\ k \geq 1\}$ family of probability laws on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Then there exists:

- Probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $\{X_k;\ k \geq 1\}$ family of independent r.v defined on Ω

Such that $\mathcal{L}(X_k) = \mu_k$.

Proof

Product space: We consider

- $(\Omega_k, \mathcal{F}_k, \mathbf{P}_k) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_k)$
- $(\Omega, \mathcal{F}, \mathbf{P}) \equiv$ product space
- $X_k(\omega) = \omega_k$ if $\omega = (\omega_k)_{k \geq 1}$

Independence: For all $k_1 < \ldots < k_n$ the r.v X_{k_j} are $\perp\!\!\!\perp$. Indeed,

$$\mathbf{P}\left(\bigcap_{j=1}^{n}(X_{k_j} \in A_j)\right) = \prod_{j=1}^{n} \mu_{k_j}(A_j) = \prod_{j=1}^{n} \mathbf{P}(X_{k_j} \in A_j),$$

This corresponds to the definition of independence.

Outline

Definition

Definition: Let $X \in \mathbb{R}^n$.
X is a Gaussian random vectors iff for all $\lambda \in \mathbb{R}^n$

$$\langle \lambda, X \rangle = \lambda^* X = \sum_{i=1}^{n} \lambda_i X_i \text{ is a real valued Gaussian r.v.}$$

Remarks:
(1) X Gaussian vector
$\Rightarrow$ Each component X_i of X is Gaussian real r.v.

(2) Key example of Gaussian vector:
Independent components $X_1, \ldots, X_n$

(3) One can easily construct an example of $X \in \mathbb{R}^2$ such that
(i) X_1, X_2 real Gaussian (ii) X is not a Gaussian vector

Characteristic function

Proposition 7.

Let X Gaussian vector, with mean m and covariance K
Then, for all $u \in \mathbb{R}^n$,

$$\mathbf{E}\left[\exp(\imath\langle u, X\rangle)\right] = e^{\imath\langle u, m\rangle - \frac{1}{2}u^* K u},$$

where u is understood as a matrix.

Proof

Random variable $\langle u, X\rangle$:

$\langle u, X\rangle$ Gaussian r.v. by assumption, with

$$\mu := \mathbf{E}[\langle u, X\rangle] = \langle u, m\rangle, \quad \text{and} \quad \sigma^2 := \mathbf{Var}(\langle u, X\rangle) = u^* K u.$$

Recall: let $Y \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$\mathbf{E}[\exp(\imath t Y)] = \exp\left(\imath t \mu - \frac{t^2}{2}\sigma^2\right), \quad t \in \mathbb{R}.$$

Gaussian moments

Proposition 8.

Let $X \sim \mathcal{N}(0,1)$. Then for all $n \in \mathbb{N}$, we have:

$$\mathbf{E}[X^n] = \begin{cases} 0 \quad \text{if } n \text{ odd}, \\ \dfrac{(2m)!}{m! 2^m}, \quad \text{if } n \text{ even}, \ n = 2m. \end{cases}$$

Affine transformations

Notation: If X Gaussian vector with mean m and covariance K
We write $X \sim \mathcal{N}(m, K)$

Proposition 9.

Let $X \sim \mathcal{N}(m_X, K_X)$, $A \in \mathbb{R}^{p,n}$ and $z \in \mathbb{R}^p$.
We set $Y = AX + z$. Then

$$Y \sim \mathcal{N}(m_Y, K_Y), \quad \text{with} \quad m_Y = z + Am_X, \quad K_Y = AK_X A^*.$$

Gaussian density

Theorem 10.

Let $X \sim \mathcal{N}(m, K)$. Then

1. X admits a density iff K is invertible.
2. Si K is invertible, the density of X is given by:

$$f(x) = \frac{1}{(2\pi)^{n/2}(\det(K))^{1/2}} \exp\left(-\frac{1}{2}(x-m)^* K^{-1}(x-m)\right).$$

Outline

Formal definition

Definition 11.

A probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is given and

- A σ-algebra $\mathcal{G} \subset \mathcal{F}$.
- $X \in \mathcal{F}$ such that $\mathbf{E}[|X|] < \infty$.

Conditional expectation of X given $\mathcal{G}$:

- Denoted by: $\mathbf{E}[X|\mathcal{G}]$
- Defined by: $\mathbf{E}[X|\mathcal{G}]$ is the r.v $Y \in L^1(\Omega)$ such that
  - (i) $Y \in \mathcal{G}$.
  - (ii) For all $A \in \mathcal{G}$, we have

$$\mathbf{E}[X\, \mathbf{1}_A] = \mathbf{E}[Y\, \mathbf{1}_A].$$

Easy examples

Example 1: If $X \in \mathcal{F}$, then $\mathbf{E}[X|\mathcal{F}] = X$.

Definition: We say that $X \perp\!\!\!\perp \mathcal{F}$ si $\sigma(X) \perp\!\!\!\perp \mathcal{F}$
$\hookrightarrow$ for all $A \in \mathcal{F}$ and $B \in \mathcal{B}(\mathbb{R})$, we have

$$\mathbf{P}((X \in B) \cap A) = \mathbf{P}(X \in B)\,\mathbf{P}(A),$$

or otherwise stated $X \perp\!\!\!\perp \mathbf{1}_A$.

Example 2: If $X \perp\!\!\!\perp \mathcal{F}$, then $\mathbf{E}[X|\mathcal{F}] = \mathbf{E}[X]$.