STOCHASTIC CALCULUS - MA 598

PROBLEMS - LIST 1

1. Probability preliminaries

Problem 1. A restaurant can serve 75 meals. In practice, it has been established that 20 % of customers with a reservation do not show up.

1.1. The restaurant owner has accepted 90 reservations. What is the probability that more than 65 persons will come?

1.2. What is the maximal number of reservations which can be accepted if we wish to serve all customers with probability ≥ 0.9?

Problem 2. Let $\gamma_{a,b}$ be the function:

$$\gamma_{a,b}(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} 1_{\{x>0\}},$$

where $a, b > 0$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

2.1. Show that $\gamma_{a,b}$ is a density.

2.2. Let X a random variable with density $\gamma_{a,b}$. Check, for $\lambda > 0$:

$$\mathbf{E}[e^{-\lambda X}] = \frac{1}{(1+\lambda b)^a}, \qquad \mathbf{E}[X] = ab, \qquad VarX = ab^2.$$

2.3. Let X (resp. X') a random variable with density $\gamma_{a,b}$ (resp. $\gamma_{a',b}$). We assume X and X' independent. Show that $X + X'$ admits the density $\gamma_{a+a',b}$.

2.4. Application: Let $X_1, X_2, ..., X_n, n$ i.i.d random variables, with law $\mathcal{N}(0,1)$. Show that $X_1^2 + X_2^2 + ... + X_n^2$ is Gamma distributed.

Problem 3. Let X_1 and X_2 two independent random variables, Poisson distributed with parameter λ. Let $Y = X_1 + X_2$. Compute

$$\mathbf{P}(X_1 = i | Y).$$

Problem 4. Let (X, Y) be a couple of random variables with joint density

$$f(x,y) = 4y(x-y)\exp(-(x+y))\mathbf{1}_{0 \leq y \leq x}.$$

4.1. Compute $\mathbf{E}[X|Y]$.

4.2. Compute $\mathbf{P}(X < 1 | Y)$.

Problem 5. The classical definition for λ-system is often given in the following way: we say that $\mathcal{L}$ is a λ-system if:

(1) $\Omega \in \mathcal{L}$.
(2) If $A, B \in \mathcal{L}$ and $B \subset A$, then $A \setminus B \in \mathcal{L}$.
(3) If $(A_n)_{n \geq 1}$ is an increasing sequence of elements of $\mathcal{L}$, then $\cup_{n \geq 1} A_n \in \mathcal{L}$.

Show that this definition is equivalent to the one seen in class.

Problem 6. Let $X = \{X_t;\, t \in \mathbb{R}_+\}$ be a stochastic process such that for all $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n$, the random variables $(\delta X_{t_j t_{j+1}})_{0 \leq j \leq n-1}$ are independent. Show that for all $0 \leq s < t < \infty$, we also have δX_{st} independent of $\mathcal{F}_s^X$.

Problem 7. For $t > 0$, let $\mathcal{C}_t$ be the collection of cylindrical sets of $C([0,t];\mathbb{R})$. Specifically, $A \in \mathcal{C}_t$ if there exists $n \geq 1$, $0 \leq t_1 < \cdots < t_n \leq t$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$ such that:

$$A = \left\{ f \in C([0,t];\mathbb{R});\; f_{t_j} \in\; B_j,\; \text{for } j = 1, \ldots, n \right\}.$$

Show that $\sigma(\mathcal{C}_t) = \mathcal{B}(C([0,t];\mathbb{R}))$.

2. Gaussian vectors

Problem 8. Let A be the matrix defined by

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

8.1. Show that there exist a centered Gaussian vector G with covariance matrix A. The coordinates of G are denoted by X, Y and Z.

8.2. Is G a random variable with density? Compute the characteristic function of G.

8.3. Characterize the law of $U = X + Y + Z$.

8.4. Show that $(X - Y, X + Z)$ is a Gaussian vector.

8.5. Determine the set of random variables $\xi = aX + bY + cZ$, independent of U.

Problem 9. Let $X, Y \sim \mathcal{N}(0,1)$ be two independent random variables. For all $a \in (-1,1)$, show that:

$$\mathbf{E}\left[\exp\left(aXY\right)\right] = \mathbf{E}\left[\exp\left(\frac{a}{2}X^2\right)\right]\mathbf{E}\left[\exp\left(-\frac{a}{2}Y^2\right)\right].$$

Problem 10. Let X and Y two independent standard Gaussian random variables $\mathcal{N}(0,1)$. We set $U = X^2 + Y^2$ and $V = \frac{X}{\sqrt{U}}$. Show that U and V are independent, and compute their law.

Problem 11. The aim of this problem is to give an example of application for the multidimensional central limit theorem. Let $(Y_i;\, i \geq 1)$ be a sequence of i.i.d real valued random variable. We will denote by F common cumulative distribution function and $\hat{F}_n$ the empirical cumulative distribution function for the n-sample $(Y_1, \ldots, Y_n)$:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{Y_i \leq x\}}, \quad x \in \mathbb{R}.$$

11.1. Let x a fixed real number. Show :

- $\hat{F}_n(x)$ converges a.s. to $F(x)$, when $n \to \infty$;
- $\sqrt{n}(\hat{F}_n(x) - F(x))$ converges in law, when $n \to \infty$, to a centered Gaussian random variable with variance $F(x)(1 - F(x))$.

11.2. We will generalize this result to a multidimensional setting. Let $x_1, x_2, ..., x_d$ be a sequence of real numbers such that $x_1 < x_2 < ... < x_d$, and X_n be the random vector in $\mathbb{R}^d$, with coordinates $X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}$ where:

$$X_n^{(i)} = \mathbf{1}_{\{Y_n \leq x_i\}}; \quad 1 \leq i \leq d,$$

for all $n \geq 1$. Show that:

$$\Big(\sqrt{n}(F_n(x_1) - F(x_1)), \ldots, \sqrt{n}(F_n(x_d) - F(x_d))\Big)$$

converges in law, when $n \to \infty$, to a centered Gaussian vector for which we will compute the covariance matrix.

Problem 12. Let $X = (X_1, \ldots, X_n)$ be a centered Gaussian vector with covariance matrix Id_n.

12.1. Show that random vector $(X_1 - \bar{X}, \ldots, X_n - \bar{X})^*$ is independent of $\bar{X}$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

12.2. Deduce that the random variables $\bar{X}$ and $W = \max_{1\leq i\leq n} X_i - \min_{1\leq i\leq n} X_i$ are independent. Why is this result (somewhat) surprising?

Problem 13. Let X and Y two real valued i.i.d random variables. We assume that $\frac{X+Y}{\sqrt{2}}$ has the same law as X and Y. We also suppose that this common law commune admits a variance, denoted by σ^2.

13.1. Show that X is centered random variable.

13.2. Show that if X_1, X_2, Y_1 and Y_2 are independent random variables having the same law as X, then $\frac{1}{2}(X_1 + X_2 + Y_1 + Y_2)$ has the same law as X.

13.3. Applying the central limit theorem, show that X is a $\mathcal{N}(0, \sigma^2)$ random variable.

Problem 14. Let $X, Y \sim \mathcal{N}(0, 1)$ two independent variables.

14.1. Show that $\frac{X}{Y}$ is well-defined, and is distributed according to a Cauchy law.

14.2. If $t \geq 0$, compute $\mathbf{P}(|X| \leq t|Y|)$.

Problem 15. If (X, Y) is a centered Gaussian vector in $\mathbb{R}^2$ with $\mathbf{E}[X^2] = \mathbf{E}[Y^2] = 1$ and if $\mathbf{E}[XY] = r$ with $r \in (-1, 1)$, calculer $\mathbf{P}(XY \geq 0)$. *Hint:* one can prove and use the following claim: $(X, Y) = (X, sX + \sqrt{1 - s^2}Z)$ with $X, Z \sim \mathcal{N}(0, 1)$ independent and $s \in (0, 1)$ to be determined. Then we invoke the result shown in Problem 14.

3. Brownian motion

Problem 16. Let B be a standard Brownian motion.

16.1. Compute, for all couple (s, t), the quantities $\mathbf{E}[B_t|\mathcal{F}_s]$ and $\mathbf{E}[B_s B_t^2]$ (we do not assume $s \leq t$ here).

16.2. Compute $\mathbf{E}[B_t^2 B_s^2]$.

16.3. What is the law of $B_t + B_s$?

16.4. Compute $\mathbf{E}[\mathbf{1}_{(B_t \leq 0)}]$ and $\mathbf{E}[B_t^2 \, \mathbf{1}_{(B_t \leq 0)}]$.

16.5. Compute $\mathbf{E}[\int_0^t \mathrm{e}^{B_s} ds]$ and $\mathbf{E}[e^{\alpha B_t} \int_0^t \mathrm{e}^{\gamma B_s} ds]$ for $\alpha, \gamma > 0$.

Problem 17. For any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ and all $0 \leq u \leq t$, show that $\mathbf{E}[f(B_t)] = \mathbf{E}[f(G\sqrt{u} + B_{t-u})]$ with a random variable $G \sim \mathcal{N}(0,1)$ independent of B_{t-u}.

Problem 18. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function whose second derivative has at most exponential growth. Show that

$$\mathbf{E}[f(x + B_t)] = f(x) + \frac{1}{2}\int_0^t \mathbf{E}[f''(x + B_s)]\, ds\,.$$

Hint: One can use the following Gaussian integration by parts formula: let $N \sim \mathcal{N}(0,1)$ and $\psi \in \mathcal{C}^1$ with exponential growth. Then $\mathbf{E}[N\,\psi(N)] = \mathbf{E}[\psi'(N)]$.

Problem 19. Consider a standard Brownian motion B. For all $\lambda, \mu \in \mathbb{R}$, compute

$$\mathbf{E}\left[\left(\mu B_1 + \lambda \int_0^1 B_u du\right)^2\right].$$

Problem 20. Show that the integral $\int_0^1 \left|\frac{B_s}{s}\right|^\alpha ds$ is finite almost surely if $\alpha < 2$.

4. Gaussian processes

Problem 21. Let $(X_n,\, n \geq 1)$ a sequence of centered Gaussian random variables, converging in law to a random variable X. Show that X is also a centered Gaussian random variable. Deduce that the process $Y = \{Y_t,\, t \geq 0\}$ given by $Y_t = \int_0^t B_u du$ is Gaussian. Compute its expected value and its covariance function.

Problem 22. We define the Brownian bridge by $Z_t = B_t - tB_1$ for $0 \leq t \leq 1$.

22.1. Show that Z is a Gaussian process independent of B_1. Give its law, that is its mean and its covariance function.

22.2. Show that the process $\tilde{Z}$, with $\tilde{Z}_t = Z_{1-t}$, has the same law as Z.

22.3. Show that the process Y, with $Y_t = (1-t)B_{\frac{t}{1-t}}$, $0 < t < 1$, has the same law as Z.

5. Martingales

Problem 23. Among the following processes, what are those who enjoy the martingale property? *Hint:* use the Fubini type relation $\mathbf{E}[\int_0^t B_u du|\mathcal{F}_s] = \int_0^t \mathbf{E}[B_u|\mathcal{F}_s]\, du$.

23.1. $M_t = {B_t}^3 - 3\int_0^t B_s\, ds$?

23.2. $Z_t = {B_t}^3 - 3tB_t$?

23.3. $X_t = tB_t - \int_0^t B_s\, ds$?

23.4. $Y_t = t^2 B_t - 2\int_0^t B_s ds$?

Problem 24. Let $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1)$. Check that B is not a $\mathcal{G}_t$-martingale. *Hint:* get a contradiction, showing that if B is a $\mathcal{G}_t$-martingale, then $\mathbf{E}[B_t|B_1] = \mathbf{E}[B_s|B_1]$ for $0 \leq s, t \leq 1$.

Problem 25. Let $Z = \{Z_t,\, t \geq 0\}$ le process defined par $Z_t = B_t - \int_0^t \frac{B_s}{s}\, ds$.

25.1. Show that Z is a Gaussian process.

25.2. Compute the expected value and the covariance function of Z. Deduce that Z is a Brownian motion.

25.3. Show that Z is not a $\mathcal{F}_t^B$-martingale, where $(\mathcal{F}_t^B)$ is the natural filtration of B. *Hint:* compute $\mathbf{E}[Z_t - Z_s|\mathcal{F}_s^B]$ for $0 \leq s < t$.

25.4. Deduce that $\mathcal{F}^Z \subset \mathcal{F}^B$, but $\mathcal{F}^Z \neq \mathcal{F}^B$.

Problem 26. Let ϕ be a bounded adapted process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and M a $(\mathcal{F}_t)$-martingale. We set

$$Y_t = M_t - \int_0^t \phi_s ds, \quad t \in [0,T].$$

Prove that

$$Y_t = E\left[\int_t^T \phi_s ds + Y_T \mid \mathcal{F}_t\right], \quad t \in [0,T]. \tag{1}$$

In the other direction, if Y satisfies (1) with a bounded adapted process ϕ, show that M defined by

$$M_t = Y_t + \int_0^t \phi_s ds, \quad t \in [0,T],$$

is a martingale.

Problem 27. Let $(M_t)_{t\geq 0}$ be a square integrable $\mathcal{F}_t$-martingale (that is such that $M_t \in L^2$ for all t).

27.1. Show that $\mathbf{E}[(M_t - M_s)^2|\mathcal{F}_s] = \mathbf{E}[{M_t}^2|\mathcal{F}_s] - {M_s}^2$ for $t > s$

27.2. Deduce that $\mathbf{E}[(M_t - M_s)^2] = \mathbf{E}[{M_t}^2] - \mathbf{E}[{M_s}^2]$ for $t > s$

27.3. Consider the function Φ defined by $\Phi(t) = \mathbf{E}[{M_t}^2]$. Check that Φ is increasing.

Problem 28. Show that if M is a $\mathcal{F}_t$-martingale, it is also a martingale with respect to its natural filtration $\mathcal{G}_t = \sigma(M_s, s \leq t)$.

Problem 29. Let τ be a positive random variable defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Show that $Z_t = \mathbf{P}(\tau \leq t|\mathcal{F}_t)$ is a sub-martingale.

Problem 30. Let X be a centered process with independent increments, such that for all $n \in \mathbb{N}^*$ and any $0 < t_1 < t_2 < \ldots < t_n$, the random variables X_{t_1}, $X_{t_2} - X_{t_1}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ are independent. In addition, we assume that X is integrable, and that $(\mathcal{F}_t)$ is the natural filtration of X. Show that X is a martingale. If we further suppose that X is square integrable, show that $X_t^2 - \mathbf{E}[X_t^2]$ is also a $(\mathcal{F}_t)$-martingale.

6. Hitting times

In this section, a designates a real number and T_a is the random time defined by $T_a = \inf\{t \geq 0 : B_t = a\}$.

Problem 31. Show that T_a is a stopping time. Compute $\mathbf{E}[e^{-\lambda T_a}]$ for all $\lambda \geq 0$. Show that $\mathbf{P}(T_a < \infty) = 1$ and that $\mathbf{E}[T_a] = \infty$.

Problem 32. Prove (avoid computations) that for $b > a > 0$, the random variable $T_b - T_a$ is independent of T_a. Deduce that the process $(T_a)_{a \geq 0}$ has independent and stationary increments.

Problem 33. Let $a < 0 < b$ and $T = T_a \wedge T_b$. Compute $\mathbf{P}(T_a < T_b)$ and $\mathbf{E}[T]$.
Hint: Apply the optional sampling theorem to B_t and $B_t^2 - t$.

Problem 34. Compute $Z_t = \mathbf{P}(T_a > 1|\mathcal{F}_t)$ for $0 \leq t \leq 1$ and $a > 0$. Recall that $\sup_{u \leq t} B_u \stackrel{(d)}{=} |B_t|$.

Problem 35. Let $I = -\inf_{s \leq T_1} B_s$. Show that I has a density given by $f_I(x) = \frac{1}{(1+x)^2} \mathbf{1}_{[0,+\infty[}(x)$.
Hint: Use $\{I \leq x\} = \{T_1 < T_{-x}\}$.

Problem 36. Let $T_1 = \inf\{t \geq 0 : \ B_t = 1\}$. Use a Brownian scaling in order to show the following identities in law:

(1) $T_1 \stackrel{(d)}{=} \frac{1}{S_1^2}$, with $S_1 = \sup(B_u, u \leq 1)$;

(2) $T_a \stackrel{(d)}{=} a^2 T_1$.

7. Wiener integral

Problem 37. In this problem we consider the process X defined by $X_t = \int_0^t (\sin s)\, dB_s$.

37.1. Show that, for each $t \geq 0$, the random variable X_t is well defined.

37.2. Show that $X = (X_t)_{t \geq 0}$ is a Gaussian process. Compute its expected value and its covariance function.

37.3. Compute $\mathbf{E}[X_t|\mathcal{F}_s]$ for $s, t \geq 0$.

37.4. Show that $X_t = (\sin t) B_t - \int_0^t (\cos s) B_s\, ds$ for all $t \geq 0$.

Problem 38. Let X be the process defined on $(0, 1)$ by: $X_t = (1 - t) \int_0^t \frac{dB_s}{1-s}$.

38.1. Show that X satisfies:

$$X_0 = 0 \quad \text{and} \quad dX_t = \frac{X_t}{t-1} dt + dB_t, \quad t \in (0, 1).$$

38.2. Show that X is a Gaussian process. Compute its expected value and its covariance function.

38.3. Show that $\lim_{t \uparrow 1} X_t = 0$ in $L^2(\Omega)$.

8. Itô's formula

Problem 39. Write the following processes as Itô processes, specifying their drift and their diffusion coefficient.

(1) $X_t = {B_t}^2$
(2) $X_t = t + \exp(B_t)$;
(3) $X_t = {B_t}^3 - 3tB_t$;
(4) $X_t = (B_t + t) \exp(-B_t - t/2)$;
(5) $X_t = \exp(t/2) \sin(B_t)$.

Problem 40. Let X and Y defined by:

$$X_t = \exp\left(\int_0^t a(s) ds\right), \quad \text{et} \quad Y_t = Y_0 + \int_0^t \left[b(s) \exp\left(-\int_0^s a(u) du\right)\right] dB_s,$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are bounded functions. We set $Z_t = X_t Y_t$. Show that $dZ_t = a(t) Z_t dt + b(t) dB_t$.

Problem 41. Let Z be the process given by $Z_t = t\,X_t\,Y_t$, where X and Y are defined by:

$$dX_t = f(t)\,dt + \sigma(t)dB_t, \quad \text{and} \quad dY_t = \eta(t)dB_t.$$

Compute dZ_t.

Problem 42. Show that $Y = (Y_t)_{t\geq 0}$ defined by $Y_t = \sin(B_t) + \frac{1}{2}\int_0^t \sin(B_s)\,ds$ is a martingale. Compute its expected value and its variance.

Problem 43. Let us assume that the following system admits a solution (X, Y):

$$\begin{cases} X_t &= x + \int_0^t Y_s\,dB_s \\ Y_t &= y - \int_0^t X_s\,dB_s \end{cases}, \quad t \geq 0.$$

Show that ${X_t}^2 + {Y_t}^2 = (x^2 + y^2)e^t$ for all $t \geq 0$.

Problem 44. We define Y and Z in the following way for $t \geq 0$:

$$Y_t = \int_0^t e^s\,dB_s, \quad \text{and} \quad Z_t = \int_0^t Y_s dB_s.$$

Compute $\mathbf{E}[Z_t]$, $\mathbf{E}[Z_t^2]$ and $\mathbf{E}[Z_t Z_s]$ for $s, t \geq 0$.

Problem 45. Let σ be an adapted continuous process in $L^2(\Omega \times \mathbb{R})$, and let $X_t = \int_0^t \sigma_s\,dB_s - \frac{1}{2}\int_0^t \sigma_s^2\,ds$. We set $Y_t = \exp(X_t)$ and $Z_t = Y_t^{-1}$.

45.1. Give an explicit expression for the dynamics of Y, that is dY_t.

45.2. Show that Y is a local martingale on $[0, T]$ for all $T > 0$. If $\sigma = 1$, show that Y is a martingale on $[0, T]$ for all $T > 0$. Compute $\mathbf{E}[Y_t]$ in this case.

45.3. Compute dZ_t.

Problem 46. Let a, b, c, z be real valued constants, and let Z be the process defined by:

$$Z_t = e^{(a-c^2/2)t + cB_t}\left(z + b\int_0^t e^{-(a-c^2/2)s - cB_s} ds\right), \; t \geq 0.$$

Give a simple expression for dZ_t.

Problem 47. Let $(X_t)_{t\geq 0}$ be a process satisfying $X_t = x + \int_0^t a_s ds + \int_0^t \sigma_s dB_s$ for $t \geq 0$. In the previous formula, x is a real number, a is a continuous process satisfying $\int_0^t |a_s| ds < \infty$ for all $t \geq 0$, and σ is an adapted continuous process verifying $\int_0^t \mathbf{E}[\sigma_s^2] ds < \infty$ for all $t \geq 0$. We wish to show that if $X \equiv 0$, then $x = 0$, $a \equiv 0$ and $\sigma \equiv 0$.

47.1. Apply Itô's formula to $Y_t = \exp(-X_t^2)$.

47.2. Prove the claim.

Problem 48. Let X be an Itô process. A function s is called scale function for X if $s(X)$ is a local martingale. Determine the scale functions of the following processes:

(1) $B_t + \nu t$;
(2) $X_t = \exp(B_t + \nu t)$;
(3) $X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s$.

Problem 49. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}_b^1$ function.

49.1. Construct a function $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ (expressed as an expected value) such that, for $t \in [0,1]$, we have $\mathbf{E}[f(B_1)|\mathcal{F}_t] = \psi(t, B_t)$.

49.2. Write Itô's formula for ψ and simplify as much as possible.

49.3. Show that, for all $t \in [0,1]$ we have:

$$\mathbf{E}[f(B_1)|\mathcal{F}_t] = \mathbf{E}[f(B_1)] + \int_0^t \mathbf{E}[f'(B_1)|\mathcal{F}_s]dB_s.$$

Problem 50. Let S be the solution of: $dS_t = rS_t dt + S_t\sigma(t, S_t)dB_t$, $t \in [0,T]$, where r is a constant and where $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a function $\mathcal{C}^{1,1}$ with bounded derivatives.

50.1. Show that $\mathbf{E}[\Phi(S_T)|\mathcal{F}_t]$ is a martingale (as a function of t) for any bounded measurable function Φ.

In the sequel, we admit that $\mathbf{E}[\Phi(S_T)|\mathcal{F}_t] = \mathbf{E}[\Phi(S_T)|S_t]$ for all $t \in [0,T]$ (Markov property for S).

50.2. Let $\varphi(t,x)$ be the function defined by $\varphi(t, S_t) = \mathbf{E}[\Phi(S_T)|S_t]$ (the existence of φ is admitted). Write dZ_t with $Z_t = \varphi(t, S_t)$.

50.3. Invoking the fact that $\varphi(t, S_t)$ is a martingale, and admitting that φ is $C^{1,2}$, show that for all $t > 0$ and all $x > 0$ we have:

$$\frac{\partial \varphi}{\partial t}(t,x) + rx\frac{\partial \varphi}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 \varphi}{\partial x^2}(t,x) = 0.$$

What is the value of $\varphi(T,x)$?

Problem 51. Let B be a d-dimensional Brownian motion. We consider an open bounded subset G of $\mathbb{R}^d$, and $\tau = \inf\{t \geq 0;\ B_t \notin G\}$. We denote by K the diameter of G, i.e $K = \sup\{|x-y|;\ x, y \in G\}$.

51.1. Show that there exists $\varepsilon = \varepsilon_K \in (0,1)$ such that $\mathbf{P}_x(\tau \geq 1) \geq \varepsilon$ for all $x \in G$.

51.2. Deduce that there exists $\rho = \rho_K \in (0,1)$ such that $\mathbf{P}_x(\tau > k) \leq \rho^k$ for all $k \geq 1$ and $x \in G$.

51.3. Deduce that $\mathbf{E}_x[\tau^p] < \infty$ for all $p \geq 1$. In particular, show that $\tau < \infty$ $\mathbf{P}_x$-almost surely for all $x \in G$.

51.4. Let $\varphi \in \mathcal{C}^2(\bar{G})$ be a harmonic function, i.e such that $\Delta\varphi = 0$ sur G. Prove that $\mathbf{E}_x[\varphi(B_\tau)] = \varphi(x)$.

51.5. We now seek some harmonic functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ having the form $\varphi(x) = f(|x|^2)$ with $f : \mathbb{R} \to \mathbb{R}$.

(1) Prove that f is solution of the differential equation $f''(y) = -\frac{d}{2y}f'(y)$ for $y > 0$.
(2) Deduce the following form for radial harmonic functions:

$$\varphi(x) = \begin{cases} x & \text{si } d = 1 \\ \ln(|x|) & \text{si } d = 2 \\ |x|^{2-d} & \text{si } d \geq 3 \end{cases}$$

Problem 52. We now consider a particular case of Problem 51, that we fix the dimension $d = 1$.

52.1. Let $a < x < b$ and $\tau = \inf\{t \geq 0;\ B_t \notin (a,b)\}$. Show that

$$\mathbf{P}_x(B_\tau = a) = \frac{b-x}{b-a}, \qquad \mathbf{P}_x(B_\tau = b) = \frac{x-a}{b-a}.$$

52.2. For $x \in \mathbb{R}$ we set $T_x = \inf\{t \geq 0;\ B_t = x\}$. Prove that $\mathbf{P}_x(T_y < \infty) = 1$ for all $x, y \in \mathbb{R}$.

52.3. Let now $s > 0$ and $x, y \in \mathbb{R}$. Show that $\mathbf{P}_x(B_t = y \text{ for a } t \geq s) = 1$.

52.4. Let $\mathcal{T}_y$ be the random set given by the points such that $B_t = y$. Applying Markov's property, show that $\mathbf{P}_x(\mathcal{T}_y \text{ unbounded}) = 1$.

Problem 53. The situation of Problem 51 is now particularized to dimension $d = 2$. For $r > 0$ we set $S_r = \inf\{t \geq 0;\ |B_t| = r\}$.

53.1. Let $x \in \mathbb{R}^2$ such that $0 < r < |x| < R$. Prove that

$$\mathbf{P}_x\left(S_r < S_R\right) = \frac{\ln(R) - \ln(|x|)}{\ln(R) - \ln(r)}.$$

53.2. Invoking the same kind of arguments as in Problem 52, show that B is recurrent, that is for any couple $x, y \in \mathbb{R}^2$ and $r > 0$ we have $\mathbf{P}_x(T_{B(y,r)} < \infty) = 1$.

53.3. Whenever $x \neq 0$, show that $\mathbf{P}_x(T_0 < \infty) = 0$, i.e the 2-dimensional Brownian motion does not hit points. *Hint:* for a fixed $R > 0$ we have

$$(T_0 < S_R) \subset \bigcap_{n \geq 1} \left(S_{1/n} < S_R\right).$$

Problem 54. Consider now the case of a Brownian motion in dimension $d \geq 3$. For $r > 0$ we set $S_r = \inf\{t \geq 0;\ |B_t| = r\}$.

54.1. Let $x \in \mathbb{R}^d$ such that $|x| > r > 0$. Prove that $\mathbf{P}_x(S_r < \infty) = (r/|x|)^{d-2}$.

54.2. Let $A_n = \{|B_t| > n^{1/2} \text{ for all } t \geq S_n\}$. Show that $\mathbf{P}_x(\limsup_n A_n) = 1$ for all $x \in \mathbb{R}^d$.

54.3. Show that $\mathbf{P}_x$-almost surely we have $\lim_{t\to\infty} |B_t| = \infty$, for all $x \in \mathbb{R}^d$.

9. Geometrical Brownian motion

Problem 55. Let S satisfying the following stochastic differential equation:

$$dS_t = S_t(b\,dt + \sigma\,dB_t), \quad S_0 = 1, \tag{2}$$

where b and σ are constants. Let $\tilde{S}_t = e^{-bt} S_t$.

55.1. Show that $(\tilde{S}_t)_{t\geq 0}$ is a martingale. Deduce the value of $\mathbf{E}[S_t]$ and $\mathbf{E}[S_t|\mathcal{F}_s]$ for any couple (t, s).

55.2. Give an expression for the drift term and the diffusion coefficient of $\frac{1}{S}$.

55.3. Show that $S_t = \exp[(b - \frac{1}{2}\sigma^2)t + \sigma B_t]$ satisfies (2), and that

$$S_T = S_t \exp[(b - \frac{1}{2}\sigma^2)(T - t) + \sigma(B_T - B_t)]$$

for all $T \geq t$.

55.4. Let L be a process verifying $dL_t = -L_t \theta_t\, dB_t$ where θ_t is an adapted continuous process in $L^2(\Omega \times \mathbb{R})$. We set $Y_t = S_t L_t$. Compute dY_t.

55.5. Let ζ_t be defined by

$$d\zeta_t = -\zeta_t(r\,dt + \theta_t\,dB_t)$$

Show that $\zeta_t = L_t e^{-rt}$. Compute $d(\zeta^{-1})_t$ and then $d(S\,\zeta)_t$. How can we choose θ in such a way that ζS is a martingale?

Problem 56. Let f be a bounded measurable function and $S = (S_t)_{t\geq 0}$ be a process verifying the equation

$$dS_t = S_t(r - f_t)dt + \sigma dB_t, \quad S_0 = x \in \mathbb{R}.$$

The following questions are independent.

56.1. Show that $e^{-rt}S_t + \int_0^t f_s\, e^{-rs} S_s\, ds$ is a local martingale.

56.2. Show that

$$S_t = x\mathrm{e}^{rt - \int_0^t f_u\,du} + \sigma \int_0^t \mathrm{e}^{r(t-s) - \int_s^t f_u\,du} dB_s$$

is a possible expression for S. In the sequel, we work with this formula for S.

56.3. Compute the expected value and the variance of S_t for $t \geq 0$.

56.4. Let $T, K > 0$. Compute $\mathbf{E}[(S_T - K)_+]$ whenever f is constant.

10. Stochastic differential equations

Problem 57. Consider the stochastic differential equation

$$X_0 = x, \quad dX_t = bX_t dt + dB_t, \quad t \geq 0,$$

with $x, b \in \mathbb{R}$.

57.1. We set $Y_t = e^{-bt}X_t$. What is the stochastic differential equation verified by Y_t? Express Y_t under the form $Y_t = x + \int_0^t f(s)dB_s$, where f is a function which will be given explicitly.

57.2. Compute $\mathbf{E}[Y_t]$ and $\mathrm{Var}(Y_t)$.

57.3. Justify the fact that $\int_0^t Y_s ds$ is a Gaussian process. Compute $\mathbf{E}[\mathrm{e}^{\int_0^t Y_s ds}]$.

57.4. For $t > s$, compute $\mathbf{E}[Y_t|\mathcal{F}_s]$ and $\mathrm{Var}\,(Y_t|\mathcal{F}_s)$ and $\mathbf{E}[X_t|\mathcal{F}_s]$ and $\mathrm{Var}\,(X_t|\mathcal{F}_s)$.

Problem 58. in this problem, we consider the following stochastic differential equation:

$$X_0 = x, \quad dX_t = (a + \alpha X_t)dt + (b + \beta X_t)dB_t, \quad t \geq 0, \tag{3}$$

where a, α, b, β are 4 real constants, and where $x \in \mathbb{R}$ is the initial condition.

58.1. We first deal with the general case of equation (3).

(1) Show that (3) admits a unique solution.
(2) We set $m(t) = \mathbf{E}[X_t]$ and $M(t) = \mathbf{E}[X_t^2)$.
  (a) Show that $m(t)$ is the unique solution of the following ordinary differential equation:
  $$y' - \alpha y = a \quad \text{et} \quad y(0) = x. \tag{4}$$
  (b) Write Itô's formula for X_t^2, where X_t is solution to (3).
  (c) Deduce that $M(t)$ is the unique solution of the following ordinary differential equation:
  $$y' - (2\alpha + \beta^2)\,y = 2(a + b\beta)m + b^2 \quad \text{et} \quad y(0) = x^2 \tag{5}$$
  where m is the solution of (4).

(d) Solve (4), then (5).

58.2. Particular case #1: we consider the case $a = b = 0$.

(1) Let $(Y_t)_{t\geq 0}$ be the unique solution of equation (3) when $a = b = 0$ such that $Y_0 = 1$. Show that
$$Y_t = \exp\left\{(\alpha - \frac{1}{2}\beta^2)t + \beta B_t\right\}.$$
(2) Show that if $\alpha \geq 0$, then Y is a sub-martingale with respect to the filtration $(\mathcal{F}_t)$. Under which condition on α, do we have the martingale property for Y?

(3) Let $(Z_t)_{t\geq 0}$ be the process defined by
$$Z_t = x + (a - b\beta)\int_0^t Y_s^{-1}\, ds + b\int_0^t Y_s^{-1}\, dB_s\,.$$
Show that the solution X_t of (3) can be written as $X_t = Y_t Z_t$.

58.3. Particular case #2: we consider the case $a = \beta = 0$:
$$X_0 = x, \quad dX_t \;=\; \alpha X_t\, dt + b\, dB_t, \quad t \geq 0. \tag{6}$$
(1) Show that the unique solution of (6) can be written as
$$X_t = e^{\alpha t}\left(x + b\int_0^t e^{-\alpha s}\, dB_s\right).$$
(2) Show that X is a Gaussian process, compute its expected value and its variance.

(3) Justifify the fact that $\int_0^t X_s ds$ is a Gaussian process. Compute $\mathbf{E}\left(\exp\int_0^t X_s ds\right)$.

(4) Compute $\mathbf{E}[X_t|\mathcal{F}_s]$ and $\mathrm{Var}\,(X_t|\mathcal{F}_s)$ for $t > s$.

(5) Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function in the class $\mathcal{C}^2$. Write Itô's formula for $Z_t = \phi(X_t)$. deduce that if $\phi(x) = \int_0^x \exp(-\alpha\frac{y^2}{b^2})\, dy$, then $Z_t = b\int_0^t \exp(-\alpha\frac{X_s^2}{b^2})\, dB_s$. Is $Z = (Z_t)$ a square integrable martingale?

(6) Let λ be a fixed number.
  (a) Compute $\Phi(t,\lambda) = \mathbf{E}[e^{\lambda X_t^2}]$.
  (b) For a fixed time $t > 0$, study the martingale $s \in [0,t] \mapsto \mathbf{E}[e^{\lambda X_t^2}|\mathcal{F}_s]$.
  (c) Show that Φ is solution of a partial differential equation.
  (d) Show that
$$\Psi(t,x) = x^2 a(t) + b(t), \text{ with } a'(t) = -a(t)(2\alpha + b^2 a(t)) \text{ and } b'(t) = -b^2 a(t)\,.$$

58.4. Particular case #3: we consider the case $a = \alpha = 0$:
$$X_0 = x \quad \text{et} \quad dX_t = (b + \beta X_t)dB_t, \quad t \geq 0 \tag{7}$$
where $x \neq -\frac{b}{\beta}$. Let h be the function defined by
$$h(y) = \frac{1}{\beta}\ln\left|\frac{b + \beta y}{b + \beta x}\right|$$
for $y \neq -\frac{b}{\beta}$

(1) We set $Y_t = h(X_t)$. What is the equation satisfied by Y_t?

(2) Deduce that the solution of equation (7) can be written as:

$$X_t = \left(x + \frac{b}{\beta}\right) \exp\left(-\frac{\beta^2}{2}t + \beta B_t\right) - \frac{b}{\beta}.$$

58.5. Particular case #4: we consider the case $a = 1$ and $b = 0$. We set $Y_t = e^{-\alpha t} X_t$.

(1) What is the differential equation satisfied by Y?

(2) Compute $\mathbf{E}[X_t]$ and $\mathrm{Var}\,(X_t)$.

Problem 59. Let $f, F, g, G : \mathbb{R}_+ \to \mathbb{R}$ be bounded continuous functions. We denote by X the solution of

$$X_0 = x \quad \text{and} \quad dX_t = [f(t) + F(t)X_t]dt + [g(t) + G(t)X_t]dB_t, \quad t \geq 0,$$

and we set Y for the solution of

$$Y_0 = 1 \quad \text{and} \quad dY_t = F(t)Y_t dt + G(t)Y_t dB_t, \quad t \geq 0.$$

59.1. Give an explicit expression for Y.

59.2. Let Z be defined by:

$$Z_t = x + \int_0^t Y_s^{-1}[f(s) - G(s)g(s)]ds + \int_0^t Y_s^{-1} g(s) dB_s\,.$$

Show that $X = YZ$.

59.3. Let $m(t) = \mathbf{E}[X_t]$ and $M_t = \mathbf{E}[X_t^2]$. Show that m is the unique solution of the ordinary differential equation $y'(t) - F(t)y(t) = f(t)$, with initial condition $y(0) = x$. Deduce that

$$m(t) = \exp(\widetilde{F}(t)) \left[x + \int_0^t \exp\left(-\widetilde{F}(s) f(s)\right) ds\right],$$

where $\widetilde{F}(t) = \int_0^t F(s)ds$. Show that M is the unique solution of

$$Y'(t) - [2F(t) + G^2(t)]y(t) = 2[f(t) + g(t)G(t)]m(t) + g^2(t) \quad \text{with} \quad y(0) = x^2.$$

Problem 60. Let S_t be the solution of $dS_t = S_t\,(r\,dt + \sigma\,dB_t)$, for some fixed parameters r, σ.

60.1. Let K be a constant, and M be the process defined by:

$$M_t = \mathbf{E}\left[\left(\frac{1}{T}\int_0^T S_u\,du - K\right)_+ \middle| \mathcal{F}_t\right].$$

Prove that M is a martingale.

60.2. Show that, setting $\zeta_t = S_t^{-1}(K - \frac{1}{T}\int_0^t S_u\,du)$, we have

$$M_t = S_t\,\mathbf{E}\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du - \zeta_t\right)_+ \middle| \mathcal{F}_t\right].$$

60.3. Let Φ be the function given by:

$$\Phi(t, x) = \mathbf{E}\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du - x\right)_+\right].$$

Show that we also have

$$\Phi(t, x) = \mathbf{E}\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du - x\right)_+ \middle| \mathcal{F}_t\right],$$

and that $M_t = S_t \Phi(t, \zeta_t)$.

60.4. Write Itô's formula for M. Deduce a partial differential equation satisfied by Φ.

Problem 61. Let α be a constant and

$$dX_t = \alpha^2 X_t^2 (1 - X_t) dt + \alpha X_t (1 - X_t) dB_t, \tag{8}$$

the initial condition being given by $X_0 = x$ with $x \in (0, 1)$. We admit that X takes values in the interval $(0, 1)$ and we set $Y_t = \frac{X_t}{1 - X_t}$.

61.1. What is the stochastic differential equation satisfied by Y?

61.2. Deduce that $X_t = \frac{x \exp(\alpha B_t - \alpha^2 t/2)}{x \exp(\alpha B_t - \alpha^2 t/2) + 1 - x}$.

Problem 62. In this problem, we consider 2 equations whose solutions are Gaussian processes.

62.1. Let $N \sim \mathcal{N}(0, 1)$ be a random variable independent of B. Check that the solution of

$$dX_t = dB_t + \frac{N - X_t}{1 - t}\, dt$$

is given by $X_t = tN + (1 - t) \int_0^t \frac{dB_s}{1 - s}$. Deduce that X is a Gaussian process, and compute its expected value and its covariance.

62.2. Let W be a Brownian motion independent of B. Check that the solution of

$$dX_t = dB_t + \frac{W_t - X_t}{1 - t}\, dt$$

is given by $X_t = (1 - t) \int_0^t \frac{W_s}{(1 - s)^2}\, ds + (1 - t) \int_0^t \frac{dB_s}{1 - s}$. Deduce that X is a Gaussian process, and compute its expected value and its covariance.