Internality of autonomous differential equations

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Parametrizing solutions of differential equations

Consider two complex numbers a and b, and the very basic differential equations:

- $y' = a$
- $y' = by$

the solution sets can be parametrized as:

- $\{at + c, c \in \mathbb{C}\}$
- $\{ce^{bt}, c \in \mathbb{C}\}$

Once we have the functions at and e^{bt}, the solution sets are recovered using a complex number and an affine map.

Systems of equations

$$\begin{cases} y_1' = y_1 \\ y_2' = iy_2 \\ z' = 6 \end{cases} \Rightarrow \begin{cases} y_1 \in \{c_1 e^t, c_1 \in \mathbb{C}\} \\ y_2 \in \{c_2 e^{it}, c_2 \in \mathbb{C}\} \\ z \in \{6t + d, d \in \mathbb{C}\} \end{cases}$$

Any three solutions are algebraically independent, so we get three independent parametrizations. Three functions, e^t, e^{it} and t, are needed.

$$\begin{cases} y_1' = 2y_1 \\ y_2' = 4y_2 \\ z_1' = 3 \\ z_2' = 6 \end{cases} \Rightarrow \begin{cases} y_1 \in \{c_1 e^{2t}, c_1 \in \mathbb{C}\} \\ y_2 \in \{c_2 e^{4t}, c_2 \in \mathbb{C}\} \Rightarrow y_2 = \frac{c_2}{c_1^2} y_1^2 \\ z_1 \in \{3t + d_1, d_1 \in \mathbb{C}\} \\ z_2 \in \{6t + d_2, d_2 \in \mathbb{C}\} \Rightarrow z_2 = 2z_1 + d_2 - 2d_1 \end{cases}$$

Only two functions, e^{2t} and t, are needed to parametrize the set of solutions.

More complicated example

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases} \Rightarrow \begin{cases} y \in \{\frac{ce^t}{e^t - d}, c \in \mathbb{C}, d \in \mathbb{C}^*\} \\ z = c - y = c - \frac{ce^t}{e^t - d} \end{cases}$$

Once we have e^t, we can recover all other solutions using two rational functions! But they are not linear anymore.

This is because this system is in (non-linear) bijection with a linear system:

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases} \Rightarrow \begin{cases} u = \frac{y}{z} \\ v = y + z \end{cases} \Rightarrow \begin{cases} u' = u \\ v' = 0 \end{cases}$$

Our goal

Consider some system of differential equations of the general form:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

where $f_i \in \mathbb{C}(x_1, \cdots, x_n)$.

What we saw: it is possible for solutions of such a system to be in rational bijection with solutions of a linear system. In that case, we obtain a rational parametrization by "transferring" the linear one.

What we'll do: the converse is true! If such a system has a parametrization using rational functions, then it must be in rational bijection with a linear system (of a special form).

How a model theorist thinks about this

We want a convenient structure to work with differential equations.

The bare minimum is a differential field of characteristic zero:
a field equipped with a differential δ that is additive and satisfies Leibniz's rule $\delta(ab) = \delta(a)b + a\delta(b)$. We write $\delta(a) = a'$.

The theory of differential fields of characteristic zero has a model companion, which is the theory DCF_0 of differentially closed fields.

Concretely, we have the differential Nullstellensatz:
let K be a differentially closed field of characteristic zero. If some finite system of differential (in)equations, defined over some parameters $A \subset K$, has a solution in some differential field extension $K < L$, then it has a solution in K.

DCF_0 = possible behavior of meromorphic functions

But does a differentially closed field have anything to do with actual differential equations?

Theorem (Seidenberg)

Let (K, δ) be any countable differential field. There exists an open $U \subset \mathbb{C}$ such that (K, δ) embeds into $(\mathrm{Mer}(U), \frac{\partial}{\partial z})$.

And any field of meromorphic functions embeds into a differentially closed field.

In the rest of the talk, I will always work in some differentially closed field $\mathcal{U}$ with $\mathbb{C} < \mathcal{U}$.

I assume that $\mathbb{C} = \{x \in \mathcal{U} : x' = 0\}$.

Systems of equations and their generic points

We will care about systems of the form:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \tag{S}$$

where the $f_i \in F(x_1, \cdots, x_n)$ are rational functions over some algebraically closed $F < \mathcal{U}$.

A solution $(a_1, \cdots, a_n)$ of (S) is generic if it satisfies no non-trivial polynomial equation over F.

Internality

The system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \qquad (S)$$

is $\mathbb{C}$-internal if there are fixed generic solutions $\overline{b}_1, \cdots, \overline{b}_m$ of (S) such that for any generic solution $\overline{a}$ of (S):

$$\overline{a} \in \mathbb{C}(\overline{b}_1, \cdots, \overline{b}_m) .$$

If $\overline{a} \in \mathbb{C}(\overline{b}_1, \cdots, \overline{b}_m)^{\mathrm{alg}}$ instead, we say it is almost $\mathbb{C}$-internal.

Example

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases} \Rightarrow \begin{cases} y \in \{\frac{ce^t}{e^t - d}, c \in \mathbb{C}, d \in \mathbb{C}^*\} \\ z = c - y = c - \frac{ce^t}{e^t - d} \end{cases}$$

pick $b_1 = e^t$

First integrals reformulation

From now on, F is always an algebraically closed subfield of $\mathbb{C}$.
The system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \tag{S}$$

induces a rational vector field on affine space $\mathbb{A}^n$.
If $g \in F(x_1, \cdots, x_n)$, its Lie derivative with respect to (S) is

$$\mathcal{L}(g) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} f_i = g(y_1, \cdots, y_n)'$$

A rational first integral of the vector field is a $g \in F(x_1, \cdots, x_n)$ such that $\mathcal{L}(g) = 0$.

Fact

(S) is almost $\mathbb{C}$-internal if and only if there exists $F < K$ such that (S) has n algebraically independent first integrals defined over K.

The question

So almost $\mathbb{C}$-internal $\Leftrightarrow$ maximal number of algebraically independent first integrals, over a differential field extension

Question

Is there a criteria for internality not involving picking a field extension?

Theorem (Rosenlicht, 74)

The system

$$y' = f(y)$$

is almost $\mathbb{C}$-internal if and only if there is $g \in F(x)$ and $\lambda \in F$ such that either :

- $\mathcal{L}(g) = \frac{\partial g}{\partial x} f = 1$
- $\mathcal{L}(g) = \frac{\partial g}{\partial x} f = \lambda g.$

Our result

Theorem (Eagles-J.)

The system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases}$$

is almost $\mathbb{C}$-internal if and only if there are $g_1, \cdots, g_n \in F(x_1, \cdots, x_n)$, algebraically independent over F, and such that for all i, either:

- $\mathcal{L}(g_i) = \lambda_i g_i$ *for some* $\lambda_i \in F$, *or*
- $\mathcal{L}(g_i) = 1$.

Remarks:

- at most one g_i such that $\mathcal{L}(g_i) = 1$
- if $\lambda_i = 0$, we recover a first integral.

Galois groups

Internal systems are structured by the following classical theorem:

Theorem

If the system (S) is $\mathbb{C}$-internal, then it is acted upon faithfully by the $\mathbb{C}$-points of an algebraic group.
We call it its Galois group, denoted $\mathrm{Aut}(S)$.

Key properties:

- if Aut(S) acts transitively, we say (S) is weakly $\mathbb{C}$-orthogonal.
  - this is equivalent to having no rational first integral
- if Aut(S)) acts freely (i.e. without fixed point), we say (S) is fundamental.

Weakly orthogonal and fundamental

Fact (Kolchin, translation by Jaoui-Moosa)

If (S) is $\mathbb{C}$-internal, weakly $\mathbb{C}$-orthogonal and fundamental, then there are:

- *an algebraic group G defined over $F \cap \mathbb{C}$,*
- $g \in F(x_1, \cdots, x_n)^n$

such that g induces a bijection between generic points of (S) and generic point of a full logarithmic differential equation on G over F.

What we can do:

(A) reduce to weakly $\mathbb{C}$-orthogonal and fundamental systems

(B) control what G can appear as a binding group

(C) write concrete equations for the solution to a full logarithmic differential equation

(D) use g to obtain an explicit condition for internality

(B) Linear Galois groups are commutative

We will only need the two most basic algebraic groups:

- $G_a(\mathbb{C}) = (\mathbb{C}, +)$,
- $G_m(\mathbb{C}) = (\mathbb{C} \setminus \{0\}, \cdot)$.

Fact

Let F be a field of complex numbers and (S) be internal, weakly $\mathbb{C}$-orthogonal. If $\mathrm{Aut}(S)$ is linear, then it is isomorphic to $G_m(\mathbb{C})^k \times G_a(\mathbb{C})^l$, where $k \in \mathbb{N}$ and $l \in \{0, 1\}$.

The action of the Galois group is always faithful, and a faithful transitive action of an abelian group is always free.

$\Rightarrow$ (S) must be fundamental!

(B) The Galois group is linear

Consider:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \tag{S}$$

where $f_i \in F(x_1, \cdots, x_n)$.
We see that:

- the action of $\mathrm{Aut}(S)$ is definably isomorphic to some birational action of an algebraic group $G(\mathbb{C})$ on the affine space $\mathbb{A}^n(\mathbb{C})$
- some structure of algebraic groups $\Rightarrow$ the Galois group is linear

(C) Logarithmic differential equations

To summarize:

- F subfield of the complex numbers
- (S) $\mathbb{C}$-internal, weakly $\mathbb{C}$ orthogonal

$\Rightarrow$ (S) has a linear Galois group
$\Rightarrow$ (S) has Galois group $G_m^k \times G_a^l$, $k \in \mathbb{N}$, $l \in \{0, 1\}$
$\Rightarrow$ some $g_1, \cdots, g_n \in F(x_1, \cdots, x_n)$ inducing a bijection to the generic type of a full logarithmic differential equation on $(G_m)^k \times (G_a)^l$

Such an equation can be expressed by:

$$\begin{cases} z_1' = \lambda_1 z_1 \\ \vdots \\ z_k' = \lambda_k z_k \\ z_{k+1}' = 1 \text{ or } = \lambda_{k+1} z_{k+1} \end{cases}$$

and fullness is equivalent to the λ_i being $\mathbb{Q}$-linearly independent.

Dimension $\Rightarrow$ either $(G_m)^{n-1} \times G_a$ or $(G_m)^n$, i.e. $k + 1 = n$

(D) What the g_i's give

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \xrightarrow{g_1, \cdots, g_n} \begin{cases} z_1' = \lambda_1 z_1 \\ \vdots \\ z_{n-1}' = \lambda_{n-1} z_{n-1} \\ z_n' = 1 \text{ or } = \lambda_n z_n \end{cases}$$

So we obtain:

$$\mathcal{L}(g_i) = g_i(y_1, \cdots, y_n)' = \lambda_i g_i \text{ if } i < n$$
$$\mathcal{L}(g_n) = g_n(y_1, \cdots, y_n)' = 1 \text{ or } \lambda_n g_n$$

$\mathbb{Q}$-linear independence of the $\lambda_i \Rightarrow$ algebraic independence of the g_i

Main theorem in the weakly orthogonal case

Theorem (Eagles-J.)

Let F be an algebraically closed field of complex numbers and some $f_1, \cdots f_n \in F(x_1, \cdots, x_n)$. The system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is almost $\mathbb{C}$-internal and weakly $\mathbb{C}$-orthogonal if and only if there are $g_1, \cdots, g_n \in F(x_1, \cdots, x_n)$, algebraically independent over F, and such that for all i, either:

- $\mathcal{L}(g_i) = \lambda_i g_i$ *for some non-zero* $\lambda_i \in F$, *or*
- $\mathcal{L}(g_i) = 1$.

What about the non-weakly $\mathbb{C}$-orthogonal case?

Example

The generic type of:

$$\begin{cases} y' = \frac{yz}{y+z} \\ z' = -\frac{yz}{y+z} \end{cases}$$

is internal, and not weakly $\mathbb{C}$-orthogonal:
$(y+z)' = 0$, so $y + z \in \mathbb{C}$, which must be fixed by the binding group $\Rightarrow$ the binding group does not act transitively.

Non-weak $\mathbb{C}$-orthogonality was witnessed by a rational first integral $(y, z) \to y + z$ to $\mathbb{C}$.

In general

Consider a system:

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \end{cases} \tag{S}$$

Then there are first integrals $\pi_1, \cdots, \pi_k$ such that for any generic solution $a_1, \cdots, a_n$ the system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_n(y_1, \cdots, y_n) \\ \pi_1(y_1, \cdots, y_n) = \pi_1(a_1, \cdots, a_n) \\ \vdots \\ \pi_k(y_1, \cdots, y_n) = \pi_k(a_1, \cdots, a_n) \end{cases} \tag{$S_{\overline{\pi}(\overline{a})}$}$$

is weakly $\mathbb{C}$-orthogonal.

(A) Reducing to weakly $\mathbb{C}$-orthogonal: proof idea

Let $F < \mathbb{C}$ be algebraically closed, and assume system (S) is $\mathbb{C}$-internal. Fix some generic solution $\overline{a}$.
The system $S_{\overline{\pi}(\overline{a})}$ is weakly $\mathbb{C}$-internal and $\mathbb{C}$-internal.

- Aut(S) is linear $\xrightarrow{\text{Galois theory}}$ Aut$(S_{\overline{\pi}(\overline{a})})$ is also linear
- $S_{\overline{\pi}(\overline{a})}$ is over over complex parameters. We can (modulo technicalities) apply our previous theorem to get g_i with Lie derivatives satisfying the desired equations
- add the rational first integrals given by the π_i

(A) Reducing to weakly $\mathbb{C}$-orthogonal: proof idea

Let $F < \mathbb{C}$ be algebraically closed, and assume system (S) is $\mathbb{C}$-internal. Fix some generic solution $\overline{a}$.
The system $S_{\overline{\pi(\overline{a})}}$ is weakly $\mathbb{C}$-internal and $\mathbb{C}$-internal.

- $\mathrm{Aut}(S)$ is linear $\xrightarrow{\text{Galois theory}}$ $\mathrm{Aut}(S_{\overline{\pi(\overline{a})}})$ is also linear
- $S_{\overline{\pi(\overline{a})}}$ is over constant parameters. We can (modulo technicalities) apply our previous theorem to get g_i with Lie derivatives satisfying the desired equations
- add the rational first integrals given by the π_i

The technicalities

Using our previous results, we obtain some $h \in F(\pi(\bar{a}))(x_1, \cdots, x_n)$ such that $\mathcal{L}(h) = \lambda h$ or 1.

We want everything to be over F!

We have that $h = \tilde{h}(x_1, \cdots, x_n, \overline{\pi}(\bar{a}))$ for some $\tilde{h} \in F(x_1, \cdots, x_n, w_1, \cdots, w_k)$
so we pick:
$g(x_1, \cdots, x_n) = \tilde{h}(x_1, \cdots, x_n, \overline{\pi}(x_1, \cdots, x_n)) \in F(x_1, \cdots, x_n)$.

To show $\lambda \in F$:

$\lambda \notin F \Rightarrow g$ maps to the generic points of the system

$$\begin{cases} \lambda' = 0 \\ z' = \lambda z \end{cases}$$

which is not almost $\mathbb{C}$-internal. But it has to be if (S) is, contradiction.

Main theorem

Theorem (Eagles-J.)

Let F be an algebraically closed field of complex numbers and some $f_1, \cdots f_n \in F(x_1, \cdots, x_n)$. The system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is almost $\mathbb{C}$-internal ~~and weakly $\mathbb{C}$-orthogonal~~ if and only if there are $g_1, \cdots, g_n \in F(x_1, \cdots, x_n)$, algebraically independent over F, and such that for all i, either:

- $\mathcal{L}(g_i) = \lambda_i g_i$ *for some ~~non-zero~~* $\lambda_i \in F$*, or*
- $\mathcal{L}(g_i) = 1$.

Orthogonality to $\mathbb{C}$

almost $\mathbb{C}$-internal = maximal number of algebraically independent rational first integral after base change

orthogonal to $\mathbb{C}$ = no rational first integral, even after base change

Theorem (Eagles-J.)

Let F be an algebraically closed field of complex numbers and some $f_1, \cdots f_n \in F(x_1, \cdots, x_n)$. The system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is orthogonal to $\mathbb{C}$ if and only if there no g such that either $\mathcal{L}(g) = \lambda g$ for some $\lambda \in F$, or $\mathcal{L}(g) = 1$.

Some consequences of orthogonality

We can construct new functions using the following operations:

- algebraic operations $+, \times, -, ^{-1}$,
- composition,
- integration,
- solving linear differential equations.

Starting with polynomials and closing under there operations, we obtain Umemura's classical functions.
They include Liouvillian functions, constructed from elementary functions using the first three operations.

Fact

If (S) is orthogonal to $\mathbb{C}$, then its generic solutions are not classical.

An application: the classic Lotka-Volterra system

The Lotka-Volterra system models predator-prey populations:

- x represents the prey population,
- y represents the predator population,

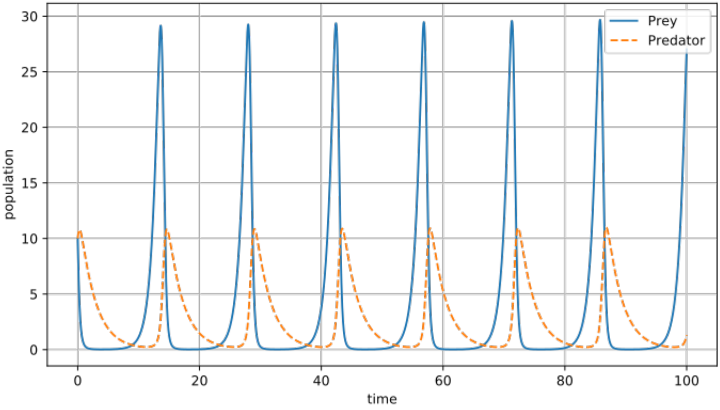
and is given by, for a, b, c, d positive real numbers:

$$\begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases}$$

We instead pick $a, b, c, d \in \mathbb{C} \setminus \{0\}$.

Graphs of (real) solutions

$$\begin{cases} x' = ax - bxy & \text{prey} \\ y' = -cy + dxy & \text{predator} \end{cases}$$



Credit: Ian Alexander (parameters, PNG version) Krishnavedala (vectorisation), from wikipedia.

Mostly not Liouvillian

Theorem (Eagles-J.)

Unless $a = c$, the generic solution of the Lotka-Volterra system:

$$\begin{cases} x' = ax + bxy \\ y' = cy + dxy \end{cases}$$

is orthogonal to the constants, and thus not Liouvillian. If $a = c$ it is elementary (proved by Varma [3]).

Enough to show that the partial differential equations:

$$c\frac{\partial g}{\partial x_0}\left(\frac{a}{c}x_0 + x_0x_1\right) + c\frac{\partial g}{\partial x_1}\left(x_1 + x_0x_1\right) = \begin{cases} 0 \\ 1 \\ \lambda g \ (\lambda \in \mathbb{Q}(a, b, c, d)^{\mathrm{alg}}) \end{cases}$$

have no rational solutions.

No rational solutions

Let $F = \mathbb{Q}(a, b, c, d)^{\mathrm{alg}}$.

The 0 and 1 cases are easy. The λg case is more complicated.

We consider the field of Laurent series:

$$K(x_0)((x_1)) = \left\{ \sum_{i=k}^{\infty} a_i x_1^i : k \in \mathbb{Z}, a_i \in F(x_0) \right\}$$

We have a differential field embedding:

$$(F(x_0, x_1), \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}) \to (F(x_0)((x_1)), \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1})$$

From our partial differential equation, we get linear differential equations (in the variable x_0) that the a_i must satisfy. There is an algorithm to find those. We obtain $\frac{a}{c} \in \mathbb{N}$.
We do the same for $K(x_1)((x_0))$ and obtain $\frac{c}{a} \in \mathbb{N}$!

Further work

- add polynomial equations between the y_j. Issue: The binding group need not be linear anymore. But the Chevalley decomposition should help.
- work over non constant parameters, for example over $\mathbb{C}(t)$.
  - an obstacle: any algebraic group can appear as a binding group by Kolchin's solution to the inverse Galois problem.
  - hope in low dimension. The case $n = 1$ has essentially been solved by Jaoui-Moosa [2]. If $n = 2$, we are interested in connected algebraic groups acting rationally on $\mathbb{P}^2$, which were classified by Enriques [1].
- can model theory say anything about parametrizations by non-rational functions? For example solutions of $y''y - (y')^2 = 0$ are $\{ce^{dx} : c, d \in \mathbb{C}\}$. The generic type is not almost $\mathbb{C}$-internal, essentially because $x \to e^x$ is not definable in DCF_0.

Thank you!

Theorem (Eagles-J.)

Let F be an algebraically closed field of complex numbers and some $f_1, \cdots f_n \in F(x_1, \cdots, x_n)$. The system

$$\begin{cases} y_1' = f_1(y_1, \cdots, y_n) \\ \vdots \\ y_n' = f_k(y_1, \cdots, y_n) \end{cases}$$

is almost $\mathbb{C}$-internal (resp. $\mathbb{C}$-orthogonal) if and only if there are $g_1, \cdots, g_n \in F(x_1, \cdots, x_n)$ (resp. no such g), algebraically independent over F such that for all i:

- $\mathcal{L}(g_i) = \lambda_i g_i$ *for some* $\lambda_i \in F$, *or*
- $\mathcal{L}(g_i) = 1$.

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