Ax-kochen ershov imaginary principle

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What is this talk about?

This talk is about elimination of imaginaries in henselian valued fields of equicharacteristic zero (joint work Rideau-Kikuchi).

1. Introduction to model theory of valued fields (AKE),
2. Introduction to elimination of imaginaries,
3. How to eliminate imaginaries in henselian valued fields? An Ax-Kochen/Ershov style strategy.
4. The results.

Valued Fields

Definition

Let K be a field and Γ and ordered abelian group. A map $v : K \to \Gamma \cup \{\infty\}$ is said to be a *valuation* if:

1. $v(x) = \infty$ if and only if $x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

An example

Consider $K = \mathbb{C}(t)$ and $\Gamma = (\mathbb{Z}, +, \leq, 0)$. Given $p(t) \in \mathbb{C}[t]$ we set:

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The extension: $v\left(\frac{p(t)}{q(t)}\right) = v(p(t)) - v(q(t))$.
number of times 0 is a zero -number of times 0 is a pole.
Example:
$v\left(\frac{2}{t}\right) = v(2) - v(t) = 0 - 1 = -1.$

Notation and terminology

Let (K, v) be a valued field. Then:

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Notation and terminology: example

Let $K = \mathbb{C}(t)$ and v the order of vanishing at 0. Then:

1. $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ is the *valuation ring*,
   In our example $\mathcal{O} = \mathbb{C}[t]_{(t)}$ i .e. quotients of polynomials whose denominator doesn't vanish at 0.
2. $\mathcal{M} = \{x \in K \mid v(x) > 0\}$ its *maximal ideal*,
   In our example $\mathcal{M} = t\mathbb{C}[t]_{(t)}$ i .e. $\{\frac{tp(t)}{q(t)} \mid \frac{p(t)}{q(t)} \in \mathcal{O}\}$.
3. $\mathbf{k} = \mathcal{O}/\mathcal{M}$ is the *residue field*,
   In our example, is isomorphic to $\mathbb{C}$.
4. Γ is the *value group*.
   In our example $(\mathbb{Z}, +, \leq, 0)$

Valued fields

Proposition

Let (K, v) be a valued field then exactly one of the following holds:

- *both K and $\mathbf{k}$ are of characteristic p (p a prime),*
- *The main filed K is of characteristic 0 while $\mathbf{k}$ is of characteristic p (example: p-adics).*
- *both K and $\mathbf{k}$ are of characteristic zero, then we say it is of equicharacteristic zero. (All the valued fields in this talk!)*

Henselian valued fields

There is a very nice subclass of valued fields!

Definition

A valued field (K, v) is said to be *henselian* if there is a *unique* extension of the valuation on K^{alg}.

Equivalently, if every non-singular zero of a polynomial over the residue field can be lifted to the main field.

Model theory of henselian valued field

SPINE PHILOSOPHY: AX-KOCHEN/ERSHOV PRINCIPLE

Theorem (Ax-Kochen/Ershov)

Let (K, k, Γ) *and* (K', k', Γ') *be two henselian valued fields of equicharacteristic zero, then* $K \equiv K'$ *if and only if* $k \equiv_{fields} k'$ *and* $\Gamma \equiv_{OAG} \Gamma'$.

Principle

The model theory of a henselian valued field of equicharacteristic zero is controlled by its residue field and its value group.

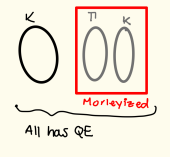
Fruitful applications of this principle

Description of the definable sets: Elimination of field quantifiers.

The quotient group $\mathrm{RV} = K^{\times}/(1+\mathcal{M})$ one has an exact sequence associated.

$$1 \to \mathbf{k}^{\times} \to \mathrm{RV} \to \Gamma \to 0$$

Pas: An equicharacteristic zero henselian valued field with an angular component eliminates field quantifiers.



More Examples:
Basarab-Kulhmann (down to the RV_n-sorts).

Interpretable sets

Definition

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Example: The projective space in the structure $(\mathbf{k}, V)$ with the $\mathcal{L}_{\mathrm{Vect}}$ *language*, where we have two sorts:

- one for the field $\mathbf{k}$ equipped with the language of rings $\mathcal{L}_{ring} = \{+, \cdot, 0, 1\}$,
- one for the vector space V with the group structure, i.e. equipped with $\mathcal{L}_G = \{0, +\}$.
- A map $\lambda : \mathbf{k} \times V \to V$ interpreted as scalar multiplication.

Example: Interpretable sets

The projective space is interpretable in $(\mathbf{k}, V)$

How do we interpret the projective space?

One can define the equivalence relation that states that two vectors lie in the same line that passes through the origin, i.e.

$$E(v, w) \text{ if and only if } \exists \ell \in \mathbf{k} \ v = \ell w = \lambda(\ell, w)$$

Elimination of imaginaries

We will denote as T a complete first order theory and $\mathfrak{M}$ its monster model.

Definition

Let T be a complete first order theory, we say that it *uniformly eliminates imaginaries* if for every $\emptyset$ definable set $D \subseteq \mathfrak{M}^n$ and $\emptyset$ definable equivalence relation E on D there is an $\emptyset$-definable function $f : D \to \mathfrak{M}^m$ such that $xEy \leftrightarrow f(x) = f(y)$.

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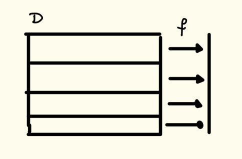

So essentially definable sets are closed under definable quotients.

An intuitive approach of elimination of imaginaries

What if a theory does not have elimination of imaginaries? What could we do?

There is an analogue of taking the Morleyization to have quantifer elimination.

Brutal approach: For each D $\emptyset$ definable set, and definable equivalence relation E on D we add a sort $S_E = D/E$ and a map $\pi : D \to S_E$ sending each element to its class.

But you might be unhappy about it...

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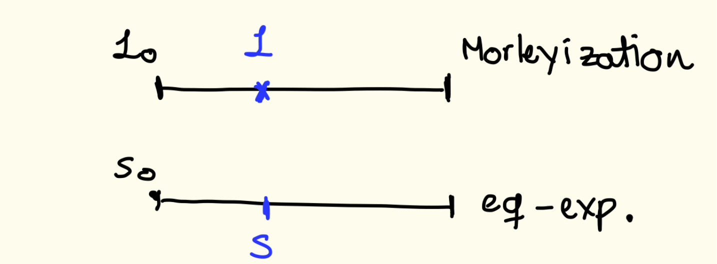
But you might be unhappy about it...
One is aiming to have a tractable description of the interpretable sets.

An intuitive approach of elimination of imaginaries

Reasonable approach: Try to find the minimal amount of sorts that are required to be consider so that we have elimination of imaginaries but the description of the definable quotients is still tractable.

Picture:



Towards an Imaginary Ax-Kochen/Ershov Principle

What was known about elimination of imaginaries in henselian valued fields?

- *ACVF* (Haskell-Hrushovski-Macpherson) down to the geometric sorts,
- p-adics and their ultraproducts (E. Hrushovski, B. Martin and S. Rideau-Kikuchi) down to the geometric sorts,
- real closed valued fields (T. Mellor) down to the geometric sorts.

Conjecture (Hrushovski, 2000)

Is there an Imaginary Ax-Kochen/ Ershov principle for henselian valued field encompassing all the previous results?

Model theory of henselian valued field

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Following the Ax-Kochen/Ershov style principle, one can set up a program in three steps for the equicharacteristic zero case:

- *First step*: assume the residue field as docile as possible and study which obstruction the value group brings to the picture [V. 2022];

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- *First step*: assume the residue field as docile as possible and study which obstruction the value group brings to the picture [V. 2022];
- *Second step*: Make the value group as tame as possible and understand the difficulties coming from the residue field [Hils,Rideau-Kikuchi 2022];

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- *First step*: assume the residue field as docile as possible and study which obstruction the value group brings to the picture [V. 2022];
- *Second step*: Make the value group as tame as possible and understand the difficulties coming from the residue field [Hils,Rideau-Kikuchi 2022];
- *Third step*: Combine the solutions of the first two steps to provide a full picture of the interpretable sets (definable quotients) in henselian valued field. [V., Rideau-Kikuchi 2023].

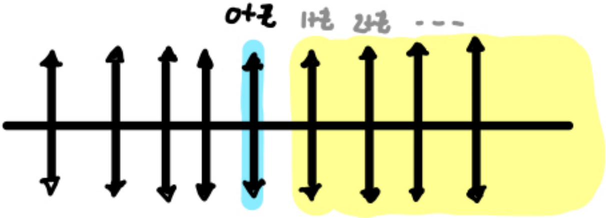
Obstructions coming from the value group

Assume the residue field as docile as possible and study which obstruction the value group brings to the picture.

Context: suppose the residue field is algebraically closed.

Easiest case *definably complete*: $(\mathbb{Q}, +, 0, \leq)$ and $(\mathbb{Z}, +, 0, \leq)$.

More general ordered abelian groups: $(\mathbb{Z}^2, \leq_{lex}, +, 0)$, which has definable convex subgroups!
Therefore, new end-segments! Not only (α, ∞) or $[\alpha, \infty)$



Obstructions coming from the value group

Key point: the complexity of the value group is reflected by increasing the class of $\mathcal{O}$-modules.

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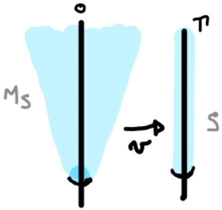

$$M_S = \{x \in K \mid v(x) \in S\}$$

Higher dimensional case: Given $M \subseteq K^n$ an $\mathcal{O}$-module,
$M \cong \oplus_{i=1}^{n} I_i$ where I_i is a fractional ideal of $\mathcal{O}$ or a copy of K.

Obstructions coming from the value group: Canonical modules and Stabilizers

New $\mathcal{O}$-modules will bring new quotients:

$$\mathrm{GL}_n(K)/\operatorname{Stab}_{(I_1,\ldots,I_n)}.$$

Given a sequence of fractional ideals $(I_1, \ldots, I_n)$ we can define a *canonical module*

$$C_{(I_1,\ldots,I_n)} := I_1 e_1 + \cdots + I_n e_n$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of K^n.

Let $\mathrm{GL}_n(K)$ be the group of invertible linear transformations and define the group,

$$\operatorname{Stab}_{(I_1,\ldots,I_n)} = \{A \in \mathrm{GL}_n(K) \mid AC_{(I_1,\ldots,I_n)} = C_{(I_1,\ldots,I_n)}\}.$$

The Stabilizer Sorts

If the residue field is algebraically closed, then these quotients are everything that you need to add!
So we take the collection of all those quotients

$$\mathrm{Mod} = \bigcup_{n\in\mathbb{N},(I_1,\ldots,I_n)} \mathrm{GL}_n(K)/\,\mathrm{Stab}_{(I_1,\ldots,I_n)}\,.$$

(this is adding codes for all the definable $\mathcal{O}$-submodules!)

Then the **stabilizer sorts are** $K \cup \mathrm{Mod}$.

Obstructions coming from the residue field

[Hils, Rideau-Kikuchi] Make the value group as tame as possible and understand the difficulties coming from the residue field.

Context: they assume the value group divisible or a $\mathbb{Z}$-group and try to answer the following question:

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How is the complexity of the residue field reflected in elimination of imaginaries in the henselian valued field?

Given a $\mathcal{O}$-lattice $s \subseteq \mathbb{K}^n$ the quotient module $s/\mathcal{M}s$ is a $\mathbf{k}$-vector space!

Once we name a basis, it is definably isomorphic to $\mathbf{k}^n$, so it is natural that it inherits the complexity of the residue field!, **but without that basis, imaginaries of $s/\mathcal{M}s$ cannot be identified with imaginaries of $\mathbf{k}$.**

The (generalized) **k**-linear imaginaries

Combing efforts!

Context: Let $M \subseteq K^n$ be an $\mathcal{O}$-module.

1. $V = M/\mathcal{M}M$ is a $\mathbf{k}$-vector space.
2. So we want to consider the two sorted structure $(\mathbf{k}, M/\mathcal{M}M)$.

The language $\mathcal{L}_{\mathrm{Vect}}$: We have two sorts:

- one for the field $\mathbf{k}$ equipped with the language of rings $\mathcal{L}_{ring} = \{+, \cdot, 0, 1\}$,
- one for the vector space V with the group structure, i.e. equipped with $\mathcal{L}_G = \{0, +\}$.
- A map $\lambda : \mathbf{k} \times V \to V$ interpreted as scalar multiplication.

The $\mathbf{k}$-linear imaginaries

1. We consider the $\mathcal{L}_{\mathrm{Vect}}$-theory of dimension ℓ vector spaces over a field.
2. For each X definable quotient of the vector space sort V in $\mathcal{L}_{\mathrm{Vect}}$. For each $M \subseteq K^n$ a $\mathcal{O}$-module:

$$X^{(\mathbf{k}, M/\mathcal{M}M)} := \text{ the interpretation of } X \text{ in the structure } (\mathbf{k}, M/\mathcal{M}M).$$

We define $T_{n,X} = \sqcup_{M \in \mathrm{Mod}} X^{(\mathbf{k}, M/\mathcal{M}M)}$.

The $\mathbf{k}$-linear imaginaries

Recall: $T_{n,X} = \sqcup_{M \in \mathrm{Mod}} X^{(\mathbf{k}, M/\mathcal{M}M)}$ where $X^{(\mathbf{k}, M/\mathcal{M}M)}$ is the interpretation of X in $(\mathbf{k}, M/\mathcal{M}M)$.

Definition (The (generalized)- $\mathbf{k}$ linear imaginaries)

$$\mathbf{k}^{leq} := \sqcup_{n,X} T_{n,X}.$$

The results

Theorem (Rideau-Kikuchi, V.)

Let M be a henselian valued field of equicharacteristic zero with angular components. Assume that the value group satisfies property **D**. *Then M (weakly) eliminates imaginaries in $K \cup \mathrm{Mod} \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

The results

Theorem (Rideau-Kikuchi, V.)

Let M be a henselian valued field of equicharacteristic zero. Assume that:

- *the value group satisfies Property* **D**,
- *one of the following conditions holds:*
  - (a) *for every $n \in \mathbb{Z}^{\geq 2}$ one has $[\Gamma : n\Gamma] < \infty$ and the pre-image in RV of any coset of $n\Gamma$ contains a point which is algebraic over $\emptyset$;*
  - (b) *or, the multiplicative group $k^{\times}$ is divisible.*

Then M has (weak) elimination of imaginaries in $K \cup \mathrm{Mod} \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.

A little about the proof

Theorem (Hrushovski)

Let T be a first order theory with home sort K (meaning that $\mathfrak{M}^{eq} = dcl^{eq}(K)$). Let $\mathcal{G}$ be some collection of sorts. Suppose that:

- *DENSITY OF DEFINABLE TYPES: For every non-empty definable set $X \subseteq K$ there is an $acl^{eq}(\ulcorner X \urcorner)$-definable type in X,*
- *CODING OF DEFINABLE TYPES: Every definable type in K^n has a code in $\mathcal{G}$ (possibly infinite). This is, if p is any (global) definable type in K^n, then the set $\ulcorner p \urcorner$ of codes of the definitions of p is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$,*

Then T weakly eliminates imaginaries down to $\mathcal{G}$.

The general strategy

Key point: density of definable types won't work if the residue field is very wild.

There are 3 steps in the proof:

1. *Step* 1*:* We show density of definable types in a reduct.
2. *Step* 2*:* We show that if a type in the reduct is A-invariant then any of its completions is $\mathrm{RV} \cup \mathrm{Lin}_{\mathrm{A}}$ invariant.
3. *Step* 3*:* We apply step 1, 2 and the fact that $\mathrm{RV} \cup \mathrm{Lin}_{\mathrm{A}}$ to show that any imaginary is in the definable closure of $\mathrm{RV} \cup \mathrm{k}^{leq}$.
4. *Step* 4*:* We break imaginaries of RV down to k^{leq} and Γ^{eq}.

The first step

Context

Let K be a henselian valued field of equicharacteristic zero and $K_1 = K^{ur}$ be its maximal unramified extension.

We work in two languages:

1. $\mathcal{L}_1$: For $K_1 = K^{ur}$
   The usual 2-sorted language (K, Γ) with Γ Morleyized.
2. $\mathcal{L}$: For the structure K
   the 3-sorted language (K, RV, Γ) with Γ Morleyized.

the first step

Theorem (Rideau-Kikuchi,V.)

Let $A = \mathrm{acl}^{eq}(A) \subseteq K^{eq}$ then for any $\mathcal{L}(A)$-definable subset $X \subseteq K^n$ there is a type $p(x) \in S^1_n(K)$ such that:

1. *$p(x) \cup X$ is consistent.*
2. *It is $\mathcal{L}_1(\mathcal{G}(A) \cup \Gamma^{eq}(A))$-definable.*
   (Its canonical base can be coded in $\mathcal{G} \cup \Gamma^{eq}$)

Sketch of the argument

The one-dimensional case: Let $X \subseteq K$ $\mathcal{L}(A)$-definable.

1. *Step* 1: We first find a generalized ball U that is $\mathcal{L}_1(A)$-definable and such that the generic type $\eta_U(x) \in S^1(K)$ is consistent with X.
2. *Step* 2*:* We complete it to a full definable type in $\mathcal{L}_1(A)$.

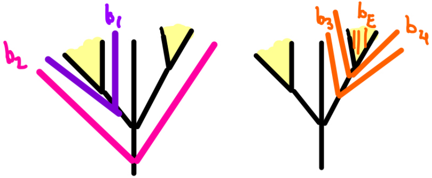
Key point: If $c \models \eta_U(x)$ then for any $a, a' \in U(K)$ one has $v(c-a) = v(c-a') = \gamma$.
Closed: unique extension.
Open: **Property** D allows us to complete the type.

Sketch of the argument: how to do step 1?

1. Let $\mathcal{B}$ be the set of closed and open balls. We define the pre-order

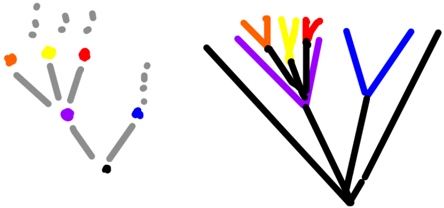
$$b_1 \trianglelefteq b_2 \text{ if and only if } b_1 \cap X \subseteq b_2 \cap X.$$

2. This is a pre-order with associated equivalence relation $\equiv$. The order $\mathcal{T}$ is a tree (remove the class of balls that don't intersect X).
3. For each class E we associate a generalized ball $b_E = \bigcap_{b \in E} b$.



Sketch of the argument: how to do step 1?

1. For each class E if $\eta_{b_E}(x)$ is not consistent with X, by compactness E has finitely many predecessors for $\trianglelefteq$, each of them in $\mathrm{acl}^{eq}(\ulcorner E \urcorner, A)$.
2. Either the statement holds or the tree has an initial discrete finitely branching tree of $\mathcal{L}_1(A)$-definable classes.



3. By 1-h-minimality. If the second case holds we can find b_E such that $b_E \cap X = b_E$. **Take its generic type!**

Step 2: Invariant extensions

How to go from an $\mathcal{L}_1$ definable type to a complete type?

Let:

$$\mathrm{Lin}_A = \bigsqcup_{s \in \mathrm{dcl}_1(A)} M/\mathcal{M}M.$$

Theorem (Rideau-Kikuchi, V.)

Let $M \prec N \models Hen_{0,0}$ sufficiently saturated and homogeneous. Let $A = \mathrm{acl}^{eq}(A) \subseteq M$. Let $a \in K(N)$ and assume that $\mathrm{tp}_1(a/M)$ *is* $\mathrm{Aut}(M/A)$*-invariant.*

Then $\mathrm{tp}(a/M)$ *is* $\mathrm{Aut}(M/ARV(M)\mathrm{Lin}_A(M))$*-invariant.*

Possible complain: $RV \cup \mathrm{Lin}_A$ is big!

Despite being big is a stably embedded set

Proposition

Let M be sufficiently saturated and homogeneous and D be a multi-sorted structure that is stably embedded. Let $e \in M$, then if e is fixed by every $\sigma \in \mathrm{Aut}(M/D(M))$ then $e \in \mathrm{dcl}(D(M))$.

In our context: $D = RV \cup \mathrm{Lin}_A$.

Weakly coding

Theorem (Rideau-Kikuchi, V.)

Let $M \vDash Hen_{0,0}$ and whose value group satisfies Property **D**. *Let $e \in M^{eq}$ and $A = \mathrm{acl}^{eq}(e)$. Then:*

$$e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A) \cup (\mathrm{RV} \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A)).$$

$\mathcal{G}'(A) = \mathcal{G}(A) \cup \Gamma^{eq}(A)$.

Weakly coding

- Let $M \models T$ sufficiently saturated and homogeneous, $e \in M^{eq}$ and $A = \mathrm{acl}^{eq}(e)$.

- There is $\mathcal{L}$-definable map g and tuple $a \in K(M)$ such that $g(a) = e$. Let $X = g^{-1}(e)$.

- **Apply step 1!**
  We can find $p \in S^1_x(M)$ such that:
  - $p \cup X$ is consistent.
  - p is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
- Take $a \models p \cup X$ then $\mathrm{tp}_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.

Weakly coding

- **Apply step** 2:
  We had found $a \vDash p \cup X$ then $\mathrm{tp}_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
  Then **The full type** $\mathrm{tp}(a/M)$- is $\mathrm{Aut}(M/\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$ invariant!

- Since $e = g(a)$ for any $\sigma \in \mathrm{Aut}(M/\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$, $\sigma(e) = e$.

- By **Fact** applied to $D = RV \cup \mathrm{Lin}_{\mathcal{G}'(A)}$ (is stably embedded!)

$$e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$$

The conclusion

- Since $e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$.
- There is h an $\mathcal{L}(\mathcal{G}'(A))$-definable function a tuple $c \in RV(M)^m \times \mathrm{Lin}^k_{\mathcal{G}'(A)}(M)$ such that $h(c) = e$.
- Take $Z = h^{-1}(e)$. This is an $\mathcal{G}'(A)$-definable set and $Z \subseteq RV \cup \mathrm{Lin}_{\mathcal{G}'(A)}$.
- Thus $\ulcorner Z \urcorner \in (RV \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A)$.
- Then $e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)\ulcorner Z \urcorner) \subseteq \mathrm{dcl}^{eq}(\mathcal{G}'(A) \cup (RV \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A))$.

Thank you!

Many thanks for your attention!