A NOTE ON THE IMAGE OF CONTINUOUS HOMOMORPHISMS OF LOCALLY PROFINITE GROUPS

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Proposition 1. *Let G be a topological group having a basis of neighbourhoods of the identity 1 consisting of open subgroups. Then for any continuous homomorphism $\rho : G \longrightarrow GL_d(\mathbb{C})$, $\ker \rho$ (=the kernel of ρ) is open.*

Proof. Since $\ker \rho$ is a subgroup of G, it suffices to show that $\ker \rho$ contains an open subgroup. From the Lemma 2("No Small Subgroup") below, there exists a neighbourhood B of the identity $e \in GL_d(\mathbb{C})$ such that the only subgroup of $GL_d(\mathbb{C})$ contained in B, is $\{e\}$. The pre-image $\rho^{-1}(B)$ is an open set in G containing 1 and then by the hypothesis on G, we get an open subgroup $U \subset \rho^{-1}(B)$. Since $\rho(U)$ is a subgroup of $GL_d(\mathbb{C})$ contained in the open subset B, it follows that $\rho(U) = e$. Hence, we have an open subgroup $U \subset \ker \rho$. □

Lemma 2 (No Small Subgroup)**.** *There exists an open set B in $GL_d(\mathbb{C})$ such that B does not contain any subgroup $\neq \{e\}$, $e =$ the identity element of $GL_d(\mathbb{C})$.*

Proof. Since the unit circle S^1 in $\mathbb{C}^\times$ ($= GL_1(\mathbb{C}) \simeq$ the center Z of $GL_d(\mathbb{C})$) is a subgroup of $GL_d(\mathbb{C})$, it is enough to verify the argument for S^1. We claim that an open set $S^{1/2} := \{z \in S^1 : Re(z) > 0\}$ in S^1 does not contain any subgroup $\neq \{e\}$. Suppose that there exists a subgroup $H \supsetneq \{e\}$ in $S^{1/2}$. Take $e^{\theta\sqrt{-1}} \in H$ for $0 \lneq \theta < \pi/2$. Let $m \in \mathbb{Z}_{>0}$ be a minimal integer such that $m\theta > \pi/2$. It then turns out that $\pi/2 < m\theta < \pi$ by the minimality. It follows that $e^{m\theta\sqrt{-1}} \notin H$. This completes the proof. □

Remark 3. 3 (a) In general, any Lie group $\mathcal{G}$ satisfies the previous Lemma 2("No Small Subgroup"). To see it, let $\mathfrak{b}$ be a small ball around 0 in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ such that the exponential map $\exp : \mathfrak{g} \longrightarrow \mathcal{G}$ induces a diffeomorphism between $\mathfrak{b}$ and its image. Set $B = \exp(\mathfrak{b}/2)$. Then for every $x \in B$ which is not the identity, there exists $m > 1$ such that $x^m \notin B$.(cf. http://mathoverflow.net/questions/61921/on-closed-totally-disconnected -subgroups-of-connected-real-lie-groups.) (b) Conversely, it is known that a locally compact, separable metric, locally connected group with no small subgroup is a Lie group. (cf. Hilbert's fifth problem.)

Now, we shall generalize Proposition 1 as follows:

Main Theorem. *Let $\mathcal{G}$ be a Lie group or $GL_d(\mathbb{C})$, and G a topological group having a basis of neighbourhoods of the identity 1 consisting of open subgroups. Then for any continuous homomorphism $\rho : G \longrightarrow \mathcal{G}$, $\ker \rho$ (=the kernel of ρ) is open.*

Proof. It is a consequence of Remark 3 and the proof of Proposition 1. □

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Example 4 (Examples for G in Main Theorem)**.** Here are examples for G satisfying the hypothesis in Main Theorem.

(a) Any discrete group G.
(b) Any profinite group G, i.e. a compact, and totally disconnected topological group, equivalently, isomorphic to the projective limit of discrete finite groups. It is known that the open compact subgroups of G form a basis of neighbourhoods of the identity (cf. [Serre's Galois Cohomology, §1. Proposition 0]). In particular, $Gal(\bar{F}/F)$ for any field F.
(c) Let $\mathbf{G}$ be a connected reductive linear(=affine) algebraic group over a local field F. Then the set $G = \mathbf{G}(F)$ of F-rational points forms a group and satisfies the hypothesis in Main Theorem. In particular, for $F = \mathbb{Q}_p$, a finite prime p and $\mathbf{G} = \mathbb{G}_m$, the multiplicative group, $\mathbf{G}(F) = \mathbb{Q}_p^\times \supsetneq \mathbb{Z}_p^\times \supsetneq (1+p\mathbb{Z}_p^\times) \supsetneq (1+p^2\mathbb{Z}_p^\times) \supsetneq \cdots \supsetneq \{1\}$.
(d) Any locally profinite group G, i.e. a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G. In fact, groups in (a), (b) and (c) are locally profinite.

Note. A *locally profinite group* is locally compact and totally disconnected. Conversely, it is known that a compact, totally disconnected topological group is *profinite.*(see [Serre's Galois Cohomology, §1. Proposition 0]). Likewise, a locally compact, totally disconnected group is *locally profinite.*(cf. [Bushnell-Henniart's The Local Langlands Conjecture for GL(2), §1.1]).

Note. An arbitrary topological space is said to be *profinite* if it homeomorphic to the projective limit of a sequence of finite sets. It is said to be *locally profinite* if every point possesses a profinite neighbourhood.

Corollary 5 (Corollary to Main Theorem)**.** *Let G be a locally profinite group and $\mathcal{G}$ a Lie group or $GL_d(\mathbb{C})$. Then for any continuous homomorphism $\rho : G \longrightarrow \mathcal{G}$,* $\ker\rho$ *(=the kernel of ρ) is open.*

Corollary 6. *Let G be a profinite group and $\mathcal{G}$ a Lie group or $GL_d(\mathbb{C})$. Then any continuous homomorphism $\rho : G \longrightarrow \mathcal{G}$ is finite, i.e. the image $\rho(G)$ is a finite subgroup in $\mathcal{G}$.*

Proof. The image $\rho(G)$ is isomorphic to $G/\ker(\rho)$, which is finite, since $\ker(\rho)$ is an open (normal) subgroup in the compact group G due to Corollary 5. □

Finally, we present the following corollary which would give an answer to some question in class on August 25, 2011.

Corollary 7. *Every complex Galois representation, i.e. continuous homomorphism from (any) Galois group to $GL_d(\mathbb{C})$, is finite.*