Lesson 28

The Height bound

The height bound Theorem relates the height of a perimeter minimizer to its cylindrical excess. Precisely, we show that if E is a perimeter minimizer in the cylinder $C(x_0, 4r, \nu)$, $x_0 \in \partial E$ and $e(E, x_0 4r, \nu)$ is small enough,

then $e(E, x_0, 4r, \nu)^{\frac{1}{2(n-1)}}$ controls the uniform distance of $C(x_0, r, \nu) \cap \partial E$ to the $(n-1)$-dimensional space $x_0 + \nu^\perp$. WLOG, we will prove this theorem in the case:

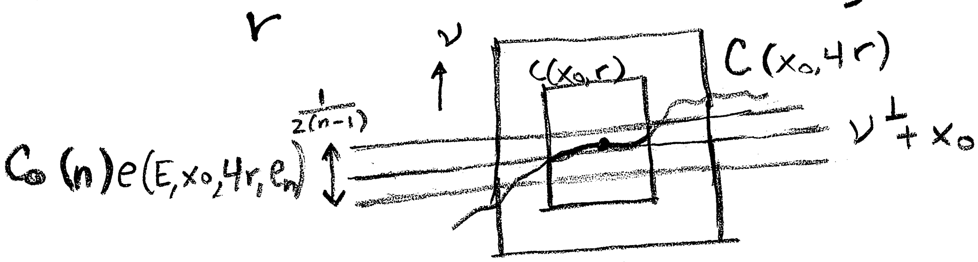
$$\nu = e_n$$

Theorem (The height bound): $n \geq 2$, there exist $\varepsilon_0(n) > 0$, $C_0(n) > 0$ such that if E minimizes perimeter in $C(x_0, 4r)$ with

$$e(E, x_0, 4r, e_n) \leq \varepsilon_0(n),$$

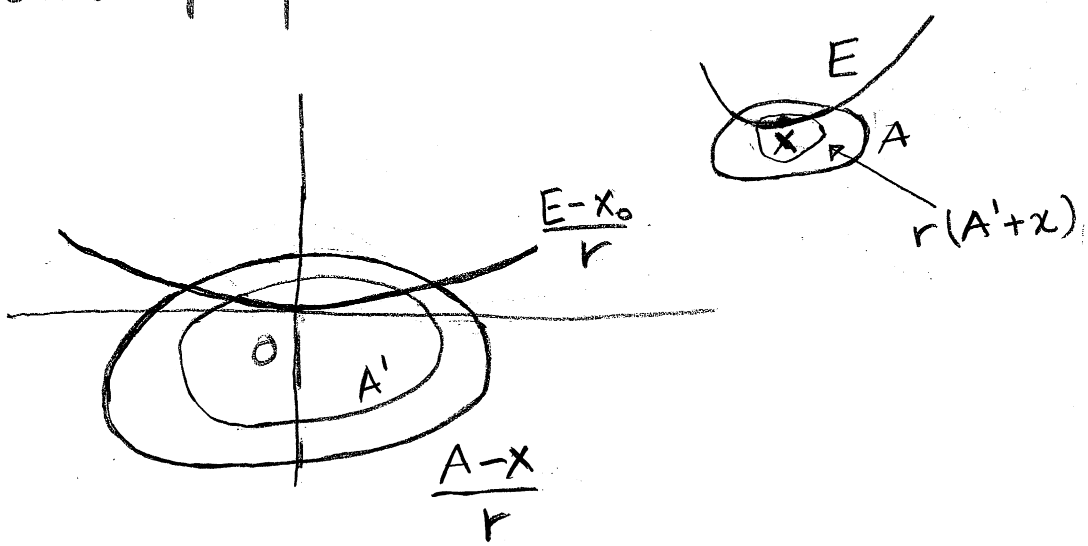
then (with q denoting the projection of $\mathbb{R}^{n-1} \times \mathbb{R}$ onto $\mathbb{R}$):

$$\sup \left\{ \frac{|qy - qx_0|}{r} : y \in C(x_0, r) \cap \partial E \right\} \leq C_0(n)\, e(E, x_0, 4r, e_n)^{\frac{1}{2(n-1)}}.$$



We first remark:

<u>Remark 1</u> : If E is a perimeter minimizer in A, then for every $x \in \mathbb{R}^n$ and $r>0$, $E_{x,r} = \frac{E-x}{r}$ is a perimeter minimizer in $A_{x,r} = \frac{A-x}{r}$. Recall our blow-up picture:



From:

$$\mu_{E_{x,r}} = \frac{(\Phi_{x,r})_{\#}\, \mu_E}{r^{n-1}}, \qquad \Phi_{x,r}(y) = \frac{y-x}{r}$$

we have; for every open set $A' \subset\subset \frac{A-x_0}{r}$

$$\int_{A'} \varphi \, d\mu_{E_{x,r}} = \frac{1}{r^{n-1}} \int_{\mathbb{R}^n} (\varphi \circ \Phi)\, d\mu_E, \quad \forall \varphi \in C_c^1(A')$$

$$= \frac{1}{r^{n-1}} \int_{r(A'+x)} \varphi\left(\frac{y-x}{r}\right) d\mu_E$$

$$= \frac{1}{r^{n-1}} \int_{r(A'+x)} \varphi_{x,r}\, d\mu_E, \quad \varphi \in C_c^1\big(r(A'+x)\big)$$

Taking the sup over all such $\varphi \in C_c^1(A')$ yields:

$$|\mu_{E_{x,r}}|(A') = \frac{1}{r^{n-1}} |\mu_E| \big(r(A'+x)\big)$$

Remark 2: Recall from previous Lectures that:

$$e(E, x, r, \nu) = e(E_{x,r}, 0, 1, \nu)$$

Proof of the Height bound Theorem:

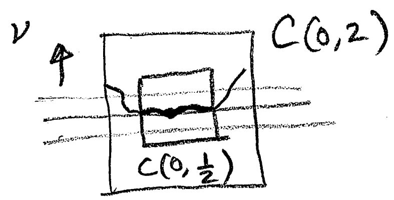
Step one: From Remark 1 and Remark 2, up to possibly replacing E with $\frac{E - x_0}{2r}$ (Note $\frac{4r}{2r} = 2$),

we can reduce to the following situation:

$$0 \in \partial E, \quad e(E, x, 2, e_n) \leq \varepsilon_0(n)$$

and we need to prove that:

$$|q x| \leq C_0(n)\, e(E, x, 2, e_n)^{\frac{1}{2(n-1)}} \quad \forall x \in C(0, \tfrac{1}{2}) \cap \partial E$$



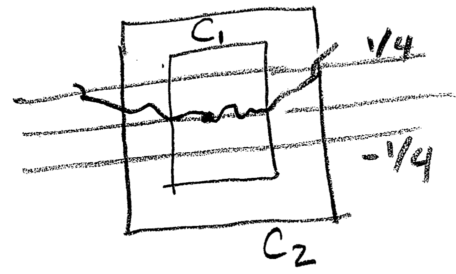
Note: $4(\frac{1}{2}) = 2$ replaces: $4(r) = 4r$.

Recall now our Small-excess position Lemma in previous Lesson. Choosing:

$$(1) \quad \boxed{\varepsilon_0(n) \leq \omega(n, \tfrac{1}{4})} \quad (t_0 = \tfrac{1}{4} \text{ in such Lemma})$$

and letting $M := C_1 \cap \partial E$, then we deduce from this Lemma that:

$$(2) \quad \boxed{|q x| \leq \tfrac{1}{4}, \quad \forall x \in M}$$



Recall that in the Excess measure Lemma in previous Lesson we showed:

$$\mathcal{H}^{n-1}(G) = \int_{M \cap p^{-1}(G)} \nu_E \cdot e_n \, d\mathcal{H}^{n-1}, \quad \forall G \text{ Borel}, \ G \subset D_1, \quad \text{and hence:}$$

$$\mathcal{H}^{n-1}(G) \leq \mathcal{H}^{n-1}(M \cap p^{-1}(G))$$

$$\Rightarrow \ 0 \leq \underbrace{\mathcal{H}^{n-1}(M \cap p^{-1}(G)) - \mathcal{H}^{n-1}(G)}_{\zeta(G)}$$

And the function ζ defines a Radon measure on D_1. Moreover,

$$0 \leq \zeta(D_1) = \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D_1) \quad ; \quad \text{recall } M = C_1 \cap \partial E$$

$$= \int_{\partial^* E \cap C_1} 1 \, d\mathcal{H}^{n-1} - \int_{\partial^* E \cap C_1} \nu_E \cdot e_n \, d\mathcal{H}^{n-1}$$

$$= \int_{\partial^* E \cap C_1} (1 - \nu_E \cdot e_n) \, d\mathcal{H}^{n-1}$$

$$= e(E, 0, 1, e_n)$$

$$\leq 2^{n-1} e(E, 0, 2, e_n) \ ;$$

since in Lesson 27 we showed:

$$e(F, x, r_1, \nu) \leq \left(\frac{r_2}{r_1}\right)^{n-1} e(F, x, r_2, \nu)$$

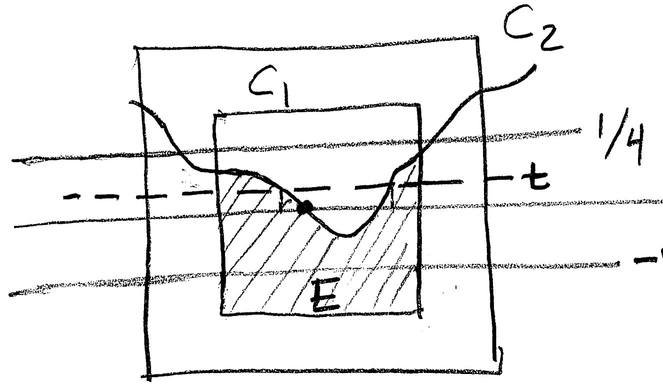
$$r_1 < r_2$$

Hence:

$$\boxed{0 \leq \mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D) \leq 2^{n-1} e(E, 0, 2, e_n)} \quad (3)$$

Recall from previous Lesson;

$$E_t := \{ z \in \mathbb{R}^{n-1} : (z,t) \in E \}$$



$M = C_1 \cap \partial E$

Note that:

$$M \cap p^{-1}(E_t) = M \cap \{ qx > t \}$$

From the discussion in previous page:

$$0 \le \underbrace{\mathcal{H}^{n-1}(M \cap p^{-1}(E_t)) - \mathcal{H}^{n-1}(D_1 \cap E_t)}_{\zeta(D_1 \cap E_t)}$$

$$= \zeta(D_1 \cap E_t)$$

$$\le \zeta(D_1) \quad ; \quad \zeta \text{ is a measure on } D_1$$

$$\lesssim 2^{n-1} e(E,0,2,e_n).$$

Hence, we have:

$$\boxed{0 \le \mathcal{H}^{n-1}(M \cap \{qx > t\}) - \mathcal{H}^{n-1}(E_t \cap D_1) \le 2^{n-1} e(E,0,2,e_n) \qquad (4)}$$

Using (1), (3), (4), the lower density estimate for perimeter minimizers:

$$(5) \qquad \boxed{\omega_{n-1} \le \frac{P(E;B(x,r))}{r^{n-1}} \le n\omega_n \qquad \forall x \in \partial E \cap C_2,\ E \text{ minimizer}}$$

and the relative isoperimetric inequality:

$$(6) \qquad \boxed{P(E;B(x,r)) \ge C(n)\,|E \cap B(x,r)|^{\frac{n-1}{n}} \qquad \forall E \subset \mathbb{R}^n \text{ set of locally finite perimeter}}$$

we now proceed to show (2).

<u>Step two</u>: Set:

$f(t) := \mathcal{H}^{n-1}(M \cap \{q_x > t\})$, decreasing

$f(t) = \mathcal{H}^{n-1}(M)$ if $t \le -\frac{1}{4}$

$f(t) = 0$ if $t \ge \frac{1}{4}$

$f(t)$ is continuous from the right

Hence $\exists\, t_0$ s.t.:

$$(*) \quad \begin{cases} f(t) \le \dfrac{\mathcal{H}^{n-1}(M)}{2}, & t \ge t_0 \\[2ex] f(t) \ge \dfrac{\mathcal{H}^{n-1}(M)}{2} & t < t_0. \end{cases}$$

<u>Claim</u>:

$$\sup\{ q\cdot x - t_0 = x_n - t_0 : x \in C_{1/2} \cap \partial E \} \leq c\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

and

$$\sup\{ t_0 - q x : x \in C_{1/2} \cap \partial E \} \leq c\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

We only need to prove the first part of the claim. The second follows by applying the same argument as in the first part to $\mathbb{R}^n \setminus E$, since E minimizer $\Rightarrow$ $\mathbb{R}^n \setminus E$ is also a minimizer.

The desired result:

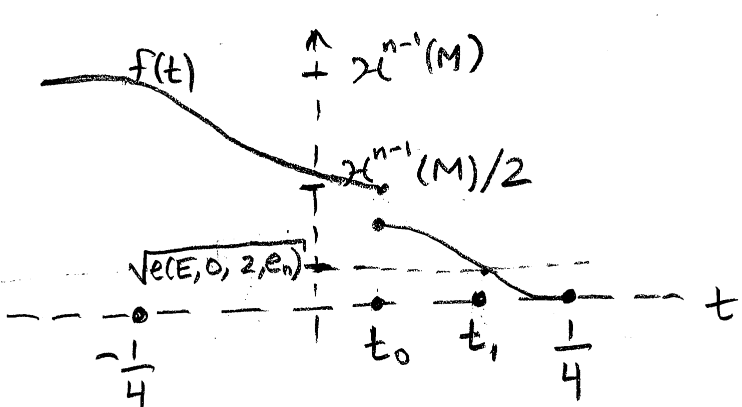
$$|qx| = |x_n| \leq C_0(n) e(E,0,2,e_n)^{\frac{1}{2(n-1)}}, \quad \forall x \in C_{\frac{1}{2}} \cap \partial E$$

follows from the claim by applying the triangle inequality.

<u>Step three</u>: Proof of the Claim:

Let $t_1 \in (t_0, \frac{1}{4})$ such that:

$$f(t) \leq \sqrt{e(E,0,2,e_n)} \quad \text{if } t \geq t_1$$

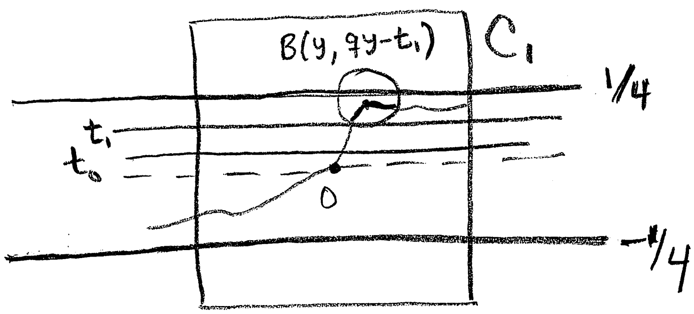


We now prove that

$$q y - t_1 \le C(n)\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}}, \quad \forall y \in C_{\frac{1}{2}} \cap \partial E.$$

Indeed, if $y \in C_{\frac{1}{2}} \cap \partial E$ and $qy > t_1$ then, since $|qy| \le \frac{1}{4}$ we have:

$$B(y, qy - t_1) \subset\subset C_2 \quad \text{with} \quad qy - t_1 < \frac{1}{2}$$



Using the lower density estimate (5) we get:

$$\omega_{n-1} (qy - t_1)^{n-1} \le P(E; B(y, qy - t_1)),$$

and since $B(y, qy - t_1) \subset C_1 \cap \{qx > t_1\}$ we have:

$$\underset{\sqrt{e(E,0,2,e_n)}}{\overset{\wedge|}{f(t_1)}} = \mathcal{H}^{n-1}(C_1 \cap \partial^* E \cap \{qx > t_1\}) \ge P(E; B(y, qy - t_1)).$$

$$\Rightarrow \quad \omega_{n-1} (qy - t_1)^{n-1} \le \sqrt{e(E,0,2,e_n)}$$

$$\Rightarrow \quad \boxed{q y - t_1 \le C(n)\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}} \quad \forall y \in C_{1/2} \cap \partial E} \quad (7)$$

Step four :

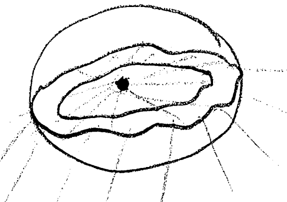
We now show that:

$$t_1 - t_0 \le c(n)\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}}$$

Notice that this inequality, together with (7) imply the claim since:

$$qx - t_0 = (qx - t_1) + (t_1 - t_0)$$

In Lecture we introduced the coarea formula for rectifiable sets;



$$\int_{\partial^* E} |\nabla^{\partial^* E} u| = \int_{-\infty}^{\infty} \mathcal{H}^{n-2}(\partial^* E \cap \{u = t\})\, dt, \quad u \text{ Lipschitz}$$

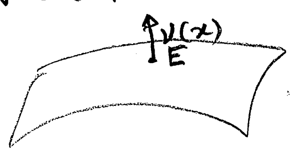
Ex:
$u = |x|$

and also:

$$\int_{\partial^* E} g\, |\nabla^{\partial^* E} u|\, d\mathcal{H}^{n-1} = \int_{-\infty}^{\infty} \int_{\partial^* E \cap \{u = t\}} g\, d\mathcal{H}^{n-2}\, dt$$

This formula is true if $\partial^* E$ is replaced by any $(n-1)$-rectifiable set

$$\nabla^{\partial^* E} u(x) = \nabla u(x) - (\nabla u(x) \cdot \nu_E(x))\, \nu_E(x) \quad \Rightarrow \quad |\nabla u(x)|^2 = |\nabla^{\partial^* E} u(x)|^2 + (\nabla u(x) \cdot \nu_E(x))^2$$



For the particular case $u(x) = qx$, $x \in \mathbb{R}^n$ we have $\nabla u(x) = e_n$ and hence $|\nabla u(x)| = 1$, which yields:

$$|\nabla^{\partial^* E} u(x)| = \sqrt{1 - (\nu_E(x) \cdot e_n)^2} = |p\, \nu_E(x)|$$

Hence:

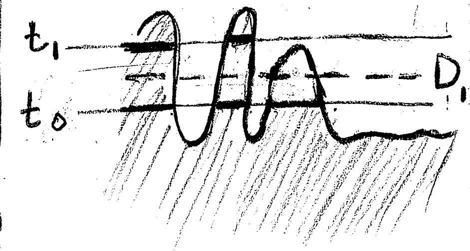
$$\int_{\partial^* E} g\, |p\, \nu_E|\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}} \int_{\partial^* E \cap \{qx = t\}} g\, d\mathcal{H}^{n-2}\, dt$$

Recall that the horizontal slice of a set $F \subset \mathbb{R}^n$ is defined as:

$$F_t = \{ z \in \mathbb{R}^{n-1} ; (z,t) \in F \}.$$

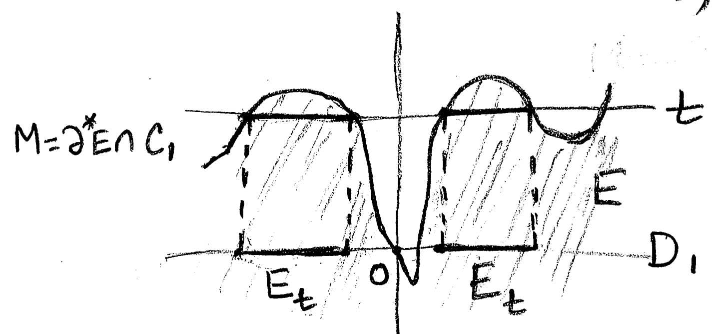
We have:

$$\boxed{\int_{\partial^* E} g \, |p\nu_E| \, d\mathcal{H}^{n-1} = \int_{\mathbb{R}} \int_{\partial^* E \cap \{qx = t\}} g \, d\mathcal{H}^{n-2} \, dt \quad (8)}$$



With this formula we can prove (see Section 18.3 in textbook):

$$\mathcal{H}^{n-2}\left(\partial^*(E_t) \Delta (\partial^* E)_t\right) = 0 \quad \text{for a.e. } t \in \mathbb{R}.$$



$C_1 = D_1 \times [-1,1]$

Going back to our proof, we have, for a.e. t:

$$\mathcal{H}^{n-2}(D_1 \cap \partial^* E_t) = \mathcal{H}^{n-2}(D_1 \cap (\partial^* E)_t) = \mathcal{H}^{n-2}((C_1 \cap \partial^* E)_t) = \mathcal{H}^{n-2}(M_t)$$

By (8) with $g = \chi_{C_1}$, and using Hölder's inequality:

$$\int_{-1}^{1} \mathcal{H}^{n-2}(D_1 \cap \partial^*(E_t))\, dt = \int_{-1}^{1} \mathcal{H}^{n-2}(C_1 \cap \partial^* E \cap \{x_n = t\})\, dt$$

$$= \int_{C_1 \cap \partial^* E} \sqrt{1 - (\nu_E(x) \cdot e_n)^2} \, d\mathcal{H}^{n-1}; \text{ by } (8)$$

$$= \int_{C_1 \cap \partial^* E} \sqrt{(1 - \nu_E(x) \cdot e_n)(1 + \nu_E(x) \cdot e_n)} \, d\mathcal{H}^{n-1}$$

$$\leq \sqrt{2} \int_{C_1 \cap \partial^* E} \sqrt{1 - \nu_E(x) \cdot e_n} \, d\mathcal{H}^{n-1} ; \quad \text{since } |\nu_E(x) \cdot e_n| \leq 1$$

$$\leq \sqrt{2\mathcal{H}^{n-1}(C_1 \cap \partial^* E)} \sqrt{\int_{C_1 \cap \partial^* E} (1 - \nu_E(x) \cdot e_n) \, d\mathcal{H}^{n-1}} ; \quad \text{by Holder's inequality}$$

$M = C_1 \cap \partial^* E$

$$= \sqrt{2\mathcal{H}^{n-1}(M)} \sqrt{\mathcal{H}^{n-1}(M) - \mathcal{H}^{n-1}(D_1)}$$

$$\leq \sqrt{2\mathcal{H}^{n-1}(M)} \; 2^{\frac{n-1}{2}} \sqrt{e(E,0,2,e_n)} ; \quad \text{by (3)}.$$

But $\mathcal{H}^{n-1}(M) \leq c(n)$ since $P(E; C(0,1)) \leq \mathcal{H}^{n-1}(\partial C(0,1))$,

and hence:

$$\boxed{\int_{-1}^{1} \mathcal{H}^{n-2}(D_1 \cap \partial^*(E_t)) \, dt \leq c(n) \sqrt{e(E,0,2,e_n)}.} \quad (9)$$

From (4), notice that for $t_0 \leq t < t_1$, the following holds:

$$\mathcal{H}^{n-1}(E_t \cap D_1) \leq \mathcal{H}^{n-1}(M \cap \{qx > t\})$$

$$= f(t)$$

$$\leq \frac{\mathcal{H}^{n-1}(M)}{2} ; \quad \text{Recall } M = \partial^* E \cap C_1, \quad t \geq t_0$$

$$\leq \frac{\mathcal{H}^{n-1}(D_1) + 2^{n-1} e(E,0,2,e_n)}{2} ; \quad \text{by (3)}.$$

$$\leq \frac{3}{4} \mathcal{H}^{n-1}(D_1) \quad \text{if } \varepsilon_0 \text{ is small enough, depending on dimension } n.$$

By applying the relative isoperimetric inequality in the ball D_1 to the set of finite perimeter $E_t \cap D_1$ yields (see (6) but n is replaced by $n-1$):

$$P(E_t, D_1) \geq c(n)\, \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}$$

$$\| $$

$$\mathcal{H}^{n-2}(\partial^*(E_t) \cap D_1)$$

Integrating and using (9):

$$\boxed{\begin{aligned} \int_{t_0}^{1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}}\, dt &\leq \tilde{c}(n) \int_{t_0}^{1} \mathcal{H}^{n-2}(\partial^*(E_t) \cap D_1)\, dt \\ &\leq \tilde{\tilde{c}}(n) \sqrt{e(E,0,2,e_n)} \qquad (10) \end{aligned}}$$

Therefore, if $t_0 \leq t \leq t_1$ and $\varepsilon_0(n)$ is small enough:

$$\mathcal{H}^{n-1}(E_t \cap D_1) \geq \mathcal{H}^{n-1}(\partial^* E \cap C_1 \cap \{q x > t\}) - 2^{n-1} e(E,0,2,e_n)$$

$$= f(t) - 2^{n-1} e(E,0,2,e_n))$$

(11) $$\geq \sqrt{e(E,0,2,e_n)} - 2^{n-1} e(E,0,2,e_n);$$ because $f(t)$ is decreasing and: $f(t) \leq \sqrt{e(E,0,2,e_n)}$, $\forall t \geq t_1$

$$\geq \frac{1}{2} \sqrt{e(E,0,2,e_n)}$$

if $\varepsilon_0(n)$ is small enough.

Indeed:

$$\sqrt{\alpha} - 2^{n-1}\alpha \geq \frac{1}{2}\sqrt{\alpha}$$

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$$\Leftrightarrow \frac{1}{2}\sqrt{\alpha} \geq 2^{n-1}\alpha$$

$$\Leftrightarrow \frac{1}{4}\alpha \geq (2^{n-1})^2\alpha^2$$

$$\Leftrightarrow \frac{1}{4} \geq (2^{n-1})^2\alpha$$

$$\Leftrightarrow \alpha \leq \frac{1}{4(2^{n-1})^2}$$

From (10) and (11):

$$\int_{t_0}^{t_1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}} dt \leq \int_{t_0}^{1} \mathcal{H}^{n-1}(E_t \cap D_1)^{\frac{n-2}{n-1}} dt \leq C(n)\sqrt{e(E,0,2,e_n)}$$

$\vee$ / by (11)

$$(t_1 - t_0)\left(\frac{1}{2}\right)^{\frac{n-2}{n-1}} \left(e(E(0,2,e_n)\right)^{\frac{n-2}{2(n-1)}}$$

$$\Rightarrow \quad (t_1 - t_2) \leq \frac{2^{\frac{n-2}{n-1}} C(n)\sqrt{e(E,0,2,e_n)}}{\sqrt{e(E,0,2,e_n)^{\frac{n-2}{n-1}}}}$$

$$= \tilde{C}(n)\sqrt{e(E,0,2,e_n)^{\frac{1}{n-1}}}; \quad \text{since } 1 - \frac{n-2}{n-1} = \frac{n-1-n+2}{n-1} = \frac{1}{n-1}$$

Hence we conclude:

$$t_1 - t \leq c(n)\, e(E,0,2,e_n)^{\frac{1}{2(n-1)}},$$

which implies our claim (see Page 28.7) and, by the triangle inequality, the Height bound Theorem as explained before. $\square$