MA 34100 Fall 2016, HW 2

September 11, 2016

1 $\sqrt{3}$ is not rational[3 pts]:

Method 1: if assuming $\sqrt{3}$ is a rational number, then it can be written as a reduced form $\sqrt{3} = \frac{p}{q}$,and p, q are relatively prime. then $p^2 = 3\ q^2$ which means $3|p^2$. Since p could be written as $p = 3m + r, r = 0, 1, 2$, then $p^2 = 9m^2 + 6mr + r^2, r^2 = 0, 1, 4$. so if $3|p^2$,then $3|r^2$ which means $r = 0$, thus $3|p$, then $\exists m, s.t.\ p = 3m$, then $9m^2 = p^2 = 3q^2 \ \Rightarrow 3m^2 = q^2$. as discussed before, we can get $3|q$, which means 3 divides p, q, which is contradictive with assumption p, q are relatively prime. Thus, $\sqrt{3}$ is not rational.

Method 2: if assuming $\sqrt{3}$ is a rational number, then it can be written as a reduced form $\sqrt{3} = \frac{p}{q}$,and p, q are relatively prime and $p^2 = 3\ q^2$. Since $p, q \in N$ means p, q has to be odd or even. If q is even then $\exists m \in N, q = 2m, \Rightarrow p^2 = 3q^2 = 3 * 4 * m^2, \Rightarrow 4|p^2 \ \ \Rightarrow 2|p$ which means p is also even. which is contradictive with assumption p, q are relatively prime. Same idea can prove p is not even, thus p, q are odd. Then $\exists m, n,\ s.t.\ p = 2m + 1,\ q = 2n + 1$. Then $p^2 = 4n^2 + 4n + 1 = 3 * (4m^2 + 4m + 1) = 12m^2 + 12m + 3 = 3q^2, \Rightarrow 2n^2 + 2n = 6m^2 + 6m + 1$. which is impossible, since left side is even, right side is odd, they can not be identity. Thus, $\sqrt{3}$ is not rational.

2 $A.8.4[2\ pts]$

Question: Prove by induction that for every $n = 1, 2, 3, ...$

$$(1 + x)^n \geq 1 + nx$$

for any $x > 0$.

Proof. For $n = 1$, since $1 + x = 1 + x$, the conclusion holds. Assume it holds for $n = k$, which means $(1 + x)^k \geq 1 + kx$. Then for $n = k + 1$, since $x > 0, (1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$, then it holds for $n = k + 1$. By principle of induction, it holds for all n. $\square$

3 $A.8.8[4\ pts]$

Question: Show that the following two principles are equivalent (i.e., assuming the validity of either one of them, prove the other).

(Principle of Induction) Let $S \subset N$ such that $1 \in S$ and for all integers n if $n \in S$, then so also is $n + 1$. Then $S = N$.
and

(Well ordering of N) If $S \subset N$ and $S \neq \emptyset$, then S has a first element (i.e., a minimal element). well ordering of N.

Proof. 1.($\Rightarrow$): Assume S has no first element. If $1 \in S$, since 1 is the first element of N, thus 1 is the first element of S, then S has a first element, which is contradictive with our assumption. Thus $1 \notin S$, let $T = N \setminus S$, then $1 \in T$ and $T \neq \emptyset$. now assume $k = n \notin S$, then $n \in T$. For $k = n+1$, since $1, 2, ..., n \notin S$, if $n+1 \in S$ then $n+1$ is the first element of S which is contradictive with our assumption. then $n+1 \notin S$ means $n+1 \in T$. Then by principle of induction, $T = N$, which means $S = N \backslash T = N \backslash N = \emptyset$,which is contradictive with $S \neq \emptyset$. Thus well ordering of N is right.

2.($\Leftarrow$): Assume $S \neq N$, then $T = N \setminus S \neq \emptyset$. By well ordering of N, if $S \neq \emptyset$, S has a first element, assume it is $k+1$. since $k+1$ is the first element of T, thus $k \notin T$, which means $k \in S$. However, by assumption if $k \in S$, then $k+1 \in S$ which is contradictive with our assumption $k+1 \notin S$. Thus, $T = \emptyset$. which means $S = N$. □

4 1.6.11[1 *pts*]

Question: Let A and B be sets of real numbers and write $C = A \cup B$. Find a relation among $supA, supB, supC$.

$supC = max\{supA, supB\}$.