MA 34100 Fall 2016, HW 3

September 21, 2016

1 1.6.23 $\cdots 2\ pts$

Let A and B be nonempty sets of real numbers and let $\delta(A,B) := inf|a-b| : a \in A, b \in B$. $\delta(A,B)$ is often called the distance between the sets A and B.
(a) Let $A = N$ and $B = R \setminus N$. Compute $\delta(A,B)$.[0.5 pts].
$\delta(A,B) = 0$, since $0 \in A$, $\{\frac{1}{n}\}_{n=2}^{\infty} \subset B$ and $0 \leq \delta(A,B) \leq \delta(\{0\}, \{\frac{1}{n}\}_{n=2}^{\infty}) = 0$.

(b) If A and B are finite sets, what does $\delta(A,B)$ represent? [0.5] pts.
$\delta(A,B) = min\{|a-b|, a \in A, b \in B\}$.

(c) Let $B = [0,1]$. What does the statement $\delta(\{x\}, B) = 0$ mean for the point x? [0.5] pts.
$x \in [0,1]$.

(d) Let $B = (0,1)$. What does the statement $\delta(\{x\}, B) = 0$ mean for the point x? [0.5] pts.
$x \in [0,1]$.

2 1.7.1 $\cdots 3\ pts$

Using the archimedean theorem, prove each of the three statements that follow the proof of the archimedean theorem.
1. No matter how large a real number x is given, there is always a natural number n larger.$\cdots$ [1 pts].
If not, then $\exists x \in R, \forall n \in N, n < x$. then x is a upper bound of N which is contradictive with Archimedean theorem: set of nature number has no upper bound. Thus, $\forall x \in R, \exists n \in N$, s.t., $n > x$.

2. Given any positive number y, no matter how large, and any positive number x, no matter how small, one can add x to itself sufficiently many times so that the result exceeds y (i.e., $nx > y$ for some $n \in N$).$\cdots$ [1 pts].
since both x, y are positive, so it is equivalent to prove $\exists n \in N, s.t., n > \frac{y}{x}$. Since $\frac{x}{y} \in R$, we can prove it by using conclusion of (1).

3. Given any positive number x, no matter how small, one can always find a fraction $\frac{1}{n}$ with n a natural number that is smaller. (i.e.,so that $\frac{1}{n} < x$).$\cdots$ [1 pts].
since x is positive, thus it is equivalent to prove $\exists n \in N, s.t., n > \frac{1}{x}$. we can prove it just by letting $y = 1$ in (2).

3 **1.9.6** $\cdots 2.5\ pts$

Show that the dyadic rationals (i.e., rational numbers of the form $\frac{m}{2^n}$ for $m \in Z, n \in N$) are dense.

Proof. what we need to prove is $\forall x, y \in R, x < y, \exists n, m \in N, s.t., x < \frac{m}{2^n} < y$. From (1.7.1-3), we know that $\exists n \in N, s.t., n > \frac{1}{y-x}$, since $2^n > n$,we get $2^n > \frac{1}{y-x} \Rightarrow 2^n y - 2^n x > 1$, then we know that there exists at least one integer between $2^n x, 2^n y$, let it be m, then we prove that $2^n x < m < 2^n y \Rightarrow x < \frac{m}{2^n} < y$. □

4 **2.2.8** $\cdots 2.5\ pts$

Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, x_n = \sqrt{2 + x_{n-1}}.$$

Show by induction that $x_n < 2$ for all n.

Proof. When $k = 1$, since $x_1 = \sqrt{2} < 2$, it holds. Assuming, it holds for $k = n$, which means $x_n < 2$, then for $k = n+1, x_{n+1} = \sqrt{2 + x_n} < \sqrt{2+2} = 2$. Then by principle of induction, we know that for $\forall n \in N, x_n < 2$. □