MA 34100 Fall 2016, HW 5

October 5, 2016

1 2.12.6 ⋯ [3 *pts*]

A careless student believes that the following statement is the Cauchy criterion.
For all $\epsilon > 0$ and all positive integers p there exists an integer N with the property that $|s_{n+p} - s_n| < \epsilon$ whenever $n \geq N$. Is this statement weaker, stronger, or equivalent to the Cauchy criterion?

Proof. This statement is weaker than Cauchy criterion.
If $\{s_n\}$ satisfies Cauchy criterion, means $\forall \epsilon > 0, \exists N_1$, when $n \geq N_1,\ m \geq N_1, |s_n - s_m| < \epsilon$. Then for any positive p, just choose $N = N_1$, then $\forall n \geq N$, $n + p > N$, thus $|s_{n+p} - s_n| < \epsilon$. It proves that if $\{s_n\}$ is a Cauchy series then it satisfies the condition of 2.12.6.

However, if a series satisfies the condition of 2.12.6, it may not be a Cauchy series. For example, $\{s_n = \ln n\}$, $\{s_n\}$ is a divergent series, but it is easy to check that it satisfies the condition of 2.12.6. The reason why they are not equivalent is, for Cauchy criterion, the N only depends on ϵ. However, here N depends on both ϵ and p. □

2 2.14.1 ⋯ [2 *pts*]

Let α and β be positive numbers. Show that

$$\lim_{n\to\infty} (\alpha^n + \beta^n)^{\frac{1}{n}} = \max\{\alpha, \beta\}$$

Proof. Wlog assume $\alpha = \max\{\alpha, \beta\}$, then $\alpha = (\alpha^n)^{1/n} \leq (\alpha^n + \beta^n)^{1/n} \leq (2\alpha^n)^{1/n} = (2)^{1/n}\alpha$. Then $\alpha = \lim_{n\to\infty} \alpha \leq \lim_{n\to\infty}(\alpha^n + \beta^n)^{1/n} \leq \lim_{n\to\infty}(2)^{1/n}\alpha^n = \alpha$, thus $\lim_{n\to\infty}(\alpha^n + \beta^n)^{\frac{1}{n}} = \max\{\alpha, \beta\}$. □

3 4.2.16 ⋯ [3 *pts*]

Show that there is no set which has the set Q as its set of accumulation points.

Proof.
Method 1:If suppose Q is the accumulation set of set A, then prove contradiction: irrational numbers are also A's accumulation points. $\forall a \in R \setminus Q$, and $\forall c > 0$, since Q is dense in R, thus there exists at least one rational number $p \in (a - c, a + c)$ and since $(a - c, a + c)$ is an open set and p is an interior point, then $\exists \delta > 0, s.t., (p - \delta, p + \delta) \subset (a - c, a + c)$. Since $p \in Q$ is an accumulation point of A, thus $(p - \delta, p + \delta) \cap A$ has infinity numbers. Since $(p - \delta, p + \delta) \cap A \subset (a - c, a + c) \cap A$, thus $(a - c, a + c) \cap A$ has infinity numbers. Thus $a \in A'$.

Method 2:Prove for any set A, A' is a closed set. Since Q is not closed, thus Q could not be an accumulation set. We just need to prove A'^c is an open set. If $A'^c = \emptyset$, then A' is closed, otherwise,$\forall x \in A'^c$, by definition, $\exists c > 0, s.t., (x-c, x+c) \cap A$ has finite numbers. since $(x-c, x+c)$ is an open subset in R, thus $\forall y \in (x-c, x+c)$ is an interior point which means $\exists \delta > 0, (y-\delta, y+\delta) \subset (x-c, x+c)$, then $(y-\delta, y+\delta) \cap A \subset (x-c, x+c) \cap A$ has finite numbers. Thus $y \in A'^c$, then $(x-c, x+c) \subset A'^c$ which means x is an interior point of A'^c. Thus A'^c is an open set. Then A' is a closed set. □

4 $4.4.3 \cdots [2\ pts]$

Show that an intersection of an arbitrary collection of closed sets is closed.

Proof. Let $I \subset \mathbb{N}$ be the index set. Then to prove $\cap_{i \in I} E_i$ is a closed set if $\{E_i\}_{i \in I}$ are closed.

Method 1: It is equivalent to prove $(\cap_{i \in I} E_i)^c$ is an open set. By De Morgan's law, $(\cap_{i \in I} E_i)^c = \cup_{i \in I} E_i^c$. Since E_i^c is open thus $\cup_{i \in I} E_i^c$ is open. Thus $\cap_{i \in I} E_i$ is a closed set.

Method 2: To prove $(\cap_{i \in I} E_i)' \subset \cap_{i \in I} E_i$.
If $(\cap_{i \in I} E_i)' = \emptyset$, then $(\cap_{i \in I} E_i)' \subset \cap_{i \in I} E_i$. otherwise, $\forall x \in (\cap_{i \in I} E_i)'$ means that $\forall c > 0, (x-c, x+c) \cap (\cap_{i \in I} E_i)$ has infinity points. Since $\forall i \in I, (x-c, x+c) \cap (\cap_{i \in I} E_i) \subset (x-c, x+c) \cap E_i$, thus x is an accumulation point of E_i. Since E_i is a closed set, thus $x \in E_i$. Since E_i is an arbitrary set in $\{E_i\}_{i \in I}$, thus $x \in \cap_{i \in I} E_i$. Then $(\cap_{i \in I} E_i)' \subset \cap_{i \in I} E_i$, so $\cap_{i \in I} E_i$ is a closed set. □