MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 1

TO BE HANDED IN BY 6PM WEDNESDAY 4TH SEPTEMBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let $a \in \mathbb{F}_q$. Show that the series

$$\sum_{r=0}^{\infty} \left(\frac{a}{t}\right)^r$$

converges in $\mathbb{F}_q((1/t))$, and hence deduce that $1/(t-a) \in \mathbb{F}_q((1/t))$.

A2. (i) Let $a_1, \ldots, a_n \in \mathbb{F}_q$. Show that the series

$$\sum_{r=0}^{\infty} (-1)^r \left(\frac{a_1}{t} + \ldots + \frac{a_n}{t^n}\right)^r$$

converges in $\mathbb{F}_q((1/t))$, and hence deduce that $(1 + a_1 t^{-1} + \ldots + a_n t^{-n})^{-1} \in \mathbb{F}_q((1/t))$.
(ii) Suppose that $f, g \in \mathbb{F}_q[t]$ and $g \neq 0$. Show that $f/g \in \mathbb{F}_q((1/t))$.

B3. Suppose that $\alpha \in \mathbb{F}_q((1/t)) \setminus \{0\}$, say

$$\alpha = \sum_{n=N}^{\infty} \frac{\alpha_n}{t^n},$$

with $\alpha_n \in \mathbb{F}_q$. Show that $1/\alpha \in \mathbb{F}_q((1/t))$.

B4. Let $r \geqslant 1$, and suppose that $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q((1/t))$.
(i) Show that there exist $x_0, x_1, \ldots, x_r \in \mathbb{F}_q[t]$, with $0 \leqslant \max_i \deg(x_i) \leqslant N$ $(1 \leqslant i \leqslant r)$, such that

$$|x_1\alpha_1 + \ldots + x_r\alpha_r - x_0| < q^{-rN}.$$

(ii) Show that there exist $x_0, x_1, \ldots, x_r \in \mathbb{F}_q[t]$, with $0 \leqslant \deg(x_0) \leqslant N$, such that

$$|x_0\alpha_i - x_i| < q^{-N/r} \quad (1 \leqslant i \leqslant r).$$

B5. Let $\alpha \in \mathbb{F}_q((1/t))$ and $d \in \mathbb{N}$.
(i) Show that there exists $x \in \mathbb{F}_q[t]$ with $0 \leqslant \deg(x) \leqslant d$ having the property that for some polynomial $f \in \mathbb{F}_q[t]$, one has

$$|\alpha x^d - f| < q^{-1}.$$

(ii) Prove that there exist infinitely many polynomials $x \in \mathbb{F}_q[t]$ satisfying the inequality $\|\alpha x^d\| < |x|^{-1/d}$, where in general

$$\|\theta\| = \min_{f \in \mathbb{F}_q[t]} |\theta - f|.$$

C6. (i) Show that $\mathbb{F}_q[t]$ is countable.
(ii) Show that the set of polynomials having coefficients in $\mathbb{F}_q[t]$ is countable, and hence deduce that the set of elements in $\mathbb{F}_q((1/t))$ that are algebraic over $\mathbb{F}_q(t)$ is countable.
(iii) Show that $\mathbb{F}_q((1/t))$ is uncountable, and hence deduce that $\mathbb{F}_q((1/t))$ contains infinitely many elements θ transcendental over $\mathbb{F}_q(t)$.

C7. Let $P(X)$ be a polynomial of degree $d \geqslant 2$ having coefficients in $\mathbb{F}_q[t]$ that is irreducible over $\mathbb{F}_q(t)$. Suppose that $P(\theta) = 0$ for some element $\theta \in \mathbb{F}_q((1/t))$.
(i) Prove the following analogue of Liouville's theorem: whenever $f, g \in \mathbb{F}_q[t]$ satisfy $g \neq 0$, then there is a positive number $c(\theta)$, depending at most on θ, such that

$$|\theta - f/g| \geqslant \frac{c(\theta)}{|g|^d}.$$

(ii) Show that

$$\Theta = \sum_{n=0}^{\infty} \frac{1}{t^{n!}}$$

is an element of $\mathbb{F}_q((1/t))$ transcendental over $\mathbb{F}_q(t)$.